# Markov Perfect Industry Dynamics With Many Firms

**Gabriel Weintraub** (Columbia)

**C. Lanier Benkard** (Stanford GSB)

**Benjamin Van Roy** (Stanford)

Econometrica, 2008

Presented by: **Simon Mongey**

**Sargent Reading Group - March 10th, 2015**

# Introduction

"Just over a decade ago, Ericson and Pakes (1995) introduced an approach to modeling a dynamic industry with heterogeneous firms..."

- Dynamics in product markets, collusion, predatory pricing, asymmetric information, advertising.
- Too hard to compute (Judd 1998 2006, Pakes-McGuire 2001)
- Their existence proofs failed (Doraszelski-Satterthwaite 2007)
- **This paper**: Introduce Oblivious Equilibria, an 'approximate' equilibrium concept which gets close to the generality of EP, easier to compute.

"...our hope is that OE will serve as the basis for more refined equilibrium notions that can be used to closely approximate MPE market outcomes in realistically sized industries."

# Environment

**Time** Discrete, infinite

**Agents** Mass $m$ consumers each with $Y$, discrete integer mass of firms

**Firm state** $x \in \{1, \dots, \bar{x}\} = X$ is the integer-valued quality level of a firm.

**Production** Constant marginal cost $c$

**Industry state** $\mathbf{s}_t \in \mathbb{N}^{\bar{x}}_{\geq 0} = \mathcal{S}$ gives *number* of firms at each quality level $x$ in period $t$. Element $s_{j,t}$.

**Markets** Cournot competition in spot market for goods

**Pay-offs** $\pi(x_{i,t}, \mathbf{s}_t)$ depends on quality, competitors' states, market structure, demand

**Exit** $\phi_{i,t} \sim G(\phi)$ sell-off value realised (private), can exit

**Entry** Cost $\kappa$, Poisson-entry $\lambda(\mathbf{s}_t) \in \mathbb{N}_+$, start $x^e \in X$

**Investment** $\iota_{i,t} \in \mathbb{R}_+$ at unit cost $d$,

$$x_{i,t+1} \sim F(x' | x_{i,t}, \iota_{i,t})$$

## "Timing protocols matter"

State: $\mathbf{s}_t$

1. Incumbent privately observes $\phi_{i,t}$ decides to exit $\rho_{i,t} = 1$ or invest $\iota_{i,t} \geq 0$
2. Number of entering firms $n \sim \text{Poi}\left(\lambda(\mathbf{s}_t)\right)$ determined and entrants pay $\kappa$
3. All incumbents compete on spot market and receive $\pi_{i,t}$
4. Exiting firms exit and receive $\phi_{i,t}$
5. Investment outcomes realised, entrants enter

State: $\mathbf{s}_{t+1}$

# Logit demand system (WBvR OpRes 2010)

- In period $t$ consumer $j$ receives

$$u_{ijt} = \underbrace{\theta_1 \ln(x_{it}+1) + \theta_2 \ln(Y - p_{it})}_{v(x_{it},p_{it})} + \varepsilon_{ijt}$$

- Given other firms' **qualities** and **prices**

$$\sigma(x_{it}, \mathbf{s}_t, \mathbf{p}_t) = \frac{\exp[v(x_{it}, p_{it})]}{1 + \sum_{j=1}^{\bar{x}} s_{j,t} \exp[v(x_{jt}, p_{jt})]}$$

- Under constant marginal cost $c$, period Nash $FOC(p_{i,t})$

$$Y - p_{it} - \theta_2(p_{it} - c)(1 - \sigma_{it}) = 0 \implies p_t^*$$

- Profit

$$\pi(x_{it}, \mathbf{s}_t) = m\sigma(x_{it}, \mathbf{s}_t, \mathbf{p}_t^*)(p_{it}^* - c)$$

# Markov Perfect Equilibrium - Value function

**Def** A **MP strategy** is $\mu = (\iota, \rho)$ where $\iota(x, \mathbf{s})$ and $\rho(x, \mathbf{s})$

**Def** A **MP entry rate function** is $\lambda(\mathbf{s})$ s.t. $n^e(\mathbf{s}) \sim \text{Poi}(\lambda(\mathbf{s}))$

**Def** The **MP value function** $V(x, \mathbf{s}|\mu, \mu', \lambda)$ is the expected net present value, of a firm at state $x$, before drawing $\phi$, following strategy $\mu$ when the industry state is $\mathbf{s}$, competitors follow $\mu'$ and the entry rate is given by $\lambda$

# Markov Perfect Equilibrium - Value function

**Def** A **MP strategy** is $\mu = (\iota, \rho)$ where $\iota(x, \mathbf{s})$ and $\rho(x, \mathbf{s})$

**Def** A **MP entry rate function** is $\lambda(\mathbf{s})$ s.t. $n^e(\mathbf{s}) \sim \text{Poi}(\lambda(\mathbf{s}))$

**Def** The **MP value function** $V(x, \mathbf{s}|\mu, \mu', \lambda)$ is the expected net present value, of a firm at state $x$, before drawing $\phi$, following strategy $\mu$ when the industry state is $\mathbf{s}$, competitors follow $\mu'$ and the entry rate is given by $\lambda$

$$\begin{aligned} V(x, \mathbf{s}|\mu, \mu', \lambda) &= \int_\Phi \pi(x, \mathbf{s}) + \mathbf{1}_{[\phi<\rho]} \left\{-d\iota + \mathbb{E}\left[V(x', \mathbf{s}'|\mu, \mu', \lambda)\right]\right\} \\ &\quad + \mathbf{1}_{[\phi \geq \rho]} \phi \, dG(\phi) \\ x' &\sim F(x'|x, \iota) \\ \mathbf{s}' &\sim \Gamma(\mathbf{s}, \mu', \mu, \lambda) \end{aligned}$$

# Hopenhayn-Rogerson

**Def** A **policy** is $\mu = (\iota, \rho)$ where $\iota(x, P)$ and $\rho(x, P)$

**Def** The **entry rate function** is a scalar $\lambda$ giving the deterministic, real-valued mass of entrants

$$\begin{aligned} V(x,P) &= \int_{\Phi} \pi(x,P) + \mathbf{1}_{[\phi<\rho]} \left\{ -d\iota(x,P) + \mathbb{E}\left[ V(x',P') \right] \right\} \\ &\qquad + \mathbf{1}_{[\phi \geq \rho]} \phi \, dG(\phi) \\ x' &\sim F(x'|x,\iota) \\ P' &= P \end{aligned}$$

**Equilibrium** $\{P, \lambda\}$ found by market-clearing and free-entry.

# Carvalho-Grassi

**Def** A **policy** is $\mu = \rho$ and $\rho(x, \mathbf{s})$

**Def** An **entry rate function** is a function $\lambda(\mathbf{s})$ giving the deterministic, integer-valued mass of entrants at each $\mathbf{s}$

$$\begin{aligned} V(x,\mathbf{s}) &= \int_{\Phi} \pi(x,\mathbf{s},P(\mathbf{s})) + \mathbf{1}_{[\phi<\rho]}\mathbb{E}\left[V(x',P')\right] \\ &\qquad + \mathbf{1}_{[\phi\geq\rho]}\phi \, dG(\phi) \\ x' &\sim F(x'|x) \\ \mathbf{s}' &= \Gamma(\mathbf{s},\lambda(\mathbf{s})) \end{aligned}$$

**Equilibrium** $\{P(\mathbf{s}), \lambda(\mathbf{s})\}$ found by market-clearing and free-entry.

# Markov Perfect Equilibrium

**Def** A **MPE** is a strategy $\mu$ and entry-rate function $\lambda$ satisfying

1. **Best-reponse** Incumbent firm strategies represent a best-response

$$V(x, \mathbf{s}|\mu, \mu, \lambda) = \sup_{\mu'} V(x, \mathbf{s}|\mu', \mu, \lambda) \quad \forall x \in \mathbb{N}, \forall s \in \mathcal{S}$$

2. **Entry** At each state, either entrants have zero expected discounted profits *or* entry is zero (or both):

$$\begin{aligned} \lambda(\mathbf{s}) \left[\beta \mathbb{E}\left[V(x^e, \mathbf{s}'|\mu, \lambda)\right] - \kappa\right] &= 0, \quad \forall s \in \mathcal{S} \\ \beta \mathbb{E}\left[V(x^e, \mathbf{s}'|\mu, \lambda)\right] - \kappa &\leq 0, \quad \forall s \in \mathcal{S} \\ \lambda(\mathbf{s}) &\geq 0, \quad \forall s \in \mathcal{S} \end{aligned}$$

# Oblivious equilibrium

Firms make decisions on the basis of their state $x$ and the ***long-run average industry state*** $\tilde{\mathbf{s}}$ **under** $(\mu', \lambda)$ which (by definition) is unchanging period to period.

$$\tilde{s}_j(\mu', \lambda) = \lim_{t \to \infty} \mathbb{E}\left[s_j(\mu', \lambda)\right] = \lambda \sum_{k=0}^{\infty} P_{\mu'}^{k}(x^e, x_j)$$

# Oblivious equilibrium - Value function

**Def** An **Ob strategy** is $\mu = (\iota, \rho)$ where $\iota(x)$ and $\rho(x)$

**Def** An **Ob entry rate function** is a scalar $\lambda$ s.t. $n^e \sim \text{Poi}(\lambda)$

**Def** The **Ob value function** $V(x|\mu', \mu, \lambda)$ is the expected net present value, of a firm at state $x$, before drawing $\phi$, following strategy $\mu$ when competitors follow $\mu'$ and the entry rate is given by $\lambda$.

# Oblivious equilibrium - Value function

**Def** An **Ob strategy** is $\mu = (\iota, \rho)$ where $\iota(x)$ and $\rho(x)$

**Def** An **Ob entry rate function** is a scalar $\lambda$ s.t. $n^e \sim \text{Poi}(\lambda)$

**Def** The **Ob value function** $V(x|\mu', \mu, \lambda)$ is the expected net present value, of a firm at state $x$, before drawing $\phi$, following strategy $\mu$ when competitors follow $\mu'$ and the entry rate is given by $\lambda$.

$$
\begin{aligned}
V(x|\mu, \mu', \lambda) &= \int_{\Phi} \pi(x, \tilde{\mathbf{s}}(\mu', \lambda)) + \mathbf{1}_{[\phi<\rho]} \left\{ -d\iota + \mathbb{E}\left[ V(x'|\mu, \mu', \lambda) \right] \right\} \\
&\quad + \mathbf{1}_{[\phi \geq \rho]} \phi \, dG(\phi) \\
x' &\sim F(x'|x, \iota) \\
\tilde{s}_j(\mu', \lambda) &= \lim_{t \to \infty} \mathbb{E}\left[ s_j(\mu', \lambda) \right] = \lambda \sum_{k=0}^{\infty} P^k_{\mu'}(x^e, x_j)
\end{aligned}
$$

# Oblivious Equilibrium

**Def** An **OE** is a (i) strategy $\mu$, (ii) entry-rate $\lambda$ satisfying

1. **Best-reponse** Incumbent firm strategies represent a best-response

$$V(x|\mu,\mu,\lambda) = \sup_{\mu'} V(x|\mu',\mu,\lambda) \quad \forall x \in \mathbb{N}$$

2. **Entry** At each state, either entrants have zero oblivious expected value *or* entry is zero (or both):

$$\begin{aligned} \lambda\left[\beta V(x^e|\mu,\lambda) - \kappa\right] &= 0 \\ \beta V(x^e|\mu,\lambda) - \kappa &\leq 0 \\ \lambda &\geq 0 \end{aligned}$$

- **Show** Stationary, long-run Poisson, existence, unique in computation

# Oblivious Equilibrium

When will this likely be a poor approximation?

1. When the industry is regularly in states far from $\tilde{\mathbf{s}}$
2. When the number of firms is small so that $\tilde{\mathbf{s}}$ is a poor approximation
3. When the industry is concentrated so large firms should be monitored closely

# Asymptotic results

**Def** Consider $m \to \infty$, then a sequence of OE $\tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)}$ are **Asymptotically ME** if $\forall x$

$$\lim_{m \to \infty} \mathbb{E}_{\tilde{\mu}^{(m)},\tilde{\lambda}^{(m)}} \left[ \sup_{\mu'} V^{(m)}\left(x, s_t^{(m)} \middle| \mu', \tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)}\right) - V^{(m)}\left(x, s_t^{(m)} \middle| \tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)}\right) \right] = 0$$

*“As market size increases deviations from OE strategy under $\mu$ have zero value”*

# Asymptotic results

**Def** Consider $m \to \infty$, then a sequence of OE $\tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)}$ are **Asymptotically ME** if $\forall x$

$$\lim_{m \to \infty} \mathbb{E}_{\tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)}} \left[ \sup_{\mu'} V^{(m)}\left(x, s_t^{(m)} \middle| \mu', \tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)}\right) - V^{(m)}\left(x, s_t^{(m)} \middle| \tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)}\right) \right] = 0$$

*"As market size increases deviations from OE strategy under $\mu$ have zero value"*

## Theorem 5.2 - Light-tailed condition and AME

**If** *there's a cut-off $z$ such that for all $\varepsilon, m$, the expected maximum effect of a change in the fraction of firms at each of the $x \geq z$ on the profits of other firms is less than $\varepsilon$,* **then the OE is AME**.

$$\mathbb{E}\left[ \sup_{m \in \mathbb{R}^+, y \in \mathbb{N}, n > 0} \left| \frac{d \ln \pi_m(y, s)}{df(x)} \right| \middle| x \geq z \right] < \varepsilon, \quad \forall \varepsilon$$

# In practice - Op Res (2010)

Recall, cost of investment is $d$ and utility

$$u_{ijt} = \theta_1 \ln(x_{it} + 1) + \theta_2 \ln(Y - p_{it}) + \varepsilon_{ijt}$$

**Table 2.** Comparison of MPE and OE investment (four firms, no entry and exit).

| | Parameters | | Investment | | |
|---|---|---|---|---|---|
| | $\theta_1$ | $d$ | MPE | OE | % diff. |
| 1. Low $d$ relative to $\theta_1$ | 0.10 | 0.10 | 0.752 | 0.754 | −0.26 |
| | 0.30 | 0.30 | 0.754 | 0.755 | −0.13 |
| | 0.50 | 0.50 | 0.741 | 0.742 | −0.11 |
| | 0.70 | 0.70 | 0.694 | 0.709 | −2.21 |
| | 0.85 | 0.70 | 0.748 | 0.765 | −2.19 |
| 2. High $d$ relative to $\theta_1$ | 0.15 | 0.27 | 0.192 | 0.185 | 3.54 |
| | 0.20 | 0.35 | 0.261 | 0.250 | 4.18 |
| | 0.30 | 0.55 | 0.238 | 0.216 | 9.28 |
| | 0.40 | 0.80 | 0.168 | 0.133 | 21.02 |
| | 0.50 | 1.00 | 0.195 | 0.158 | 18.62 |

*Note.* Investment simulated with a relative precision of 1.0% and a confidence level of 99%.

# Conclusion

- Approximation that works well in large industries or without entry and exit
- Stands somewhere between MPE and KS.
- Allows for 'more meaningful' competition between agents
- Extensions
  1. Concentrated large firms and oblivious fringe (Weintraub and Irfach, WP 2014)
     - Banking: Corbae and d'Erasmo (WP 2013)
  2. Partially oblivious equilibrium (Weintraub, Benkard Jeziorski, WP 2014)
  3. Non-stationary (WBJ 2008)
  4. Aggregate shocks (OpRes 2010)