SPECULATIVE INVESTOR BEHAVIOR AND LEARNING

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Sargent Reading Group
INTRODUCTION
This paper is a follow up to the Harrison Kreps 1978 paper, "Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations".

Explores speculative premia when agents are Bayesian learners.
In a given period, an individual values an asset according to

$$(1 + r)p_t = E_t[y_{t+1} + p_{t+1}]$$

If all agents are of one type, then this is just pinned down by:

$$p_t^* = E_t \left[ \sum_{\tau=1}^{\infty} \left( \frac{1}{1 + r} \right)^\tau y_{t+\tau} \right]$$

If agents differ in beliefs, you may think you can sell it to someone else at a higher price tomorrow

$$(1 + r)p_t = E_{i,t}[y_{t+1}] + E_{i,t}[p_{t+1}] > (1 + r)p_t^*$$
Refer to $p_t - p_t^*$ as the speculative premia
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Two assets:
- Risk-free asset that can be traded at $\frac{1}{r}$ and pays 1 in following period.
- Risky asset that can be bought and sold (no short selling) which pays 1 with probability $\theta$. The price at which it trades clears the market.

$J$ types of risk-neutral traders with a continuum of each type
- Agents do not know the true $\theta$; they will instead learn about it as Bayesians.
- Types differ only in the prior which they place over values of $\theta$ ($\pi_i(\theta)$).
- Assume $\pi_i$ is twice differentiable and uniformly bounded below.
Given a prior distribution of $\pi_i(\theta)$, an agent who has seen $s$ payments of the dividend in $t$ periods has a posterior given by

$$
\xi_i(\theta|x, t) = \frac{\theta^s (1 - \theta)^{t-s} \pi_i(\theta)}{\int_{\zeta=0}^{1} \zeta^s (1 - \zeta)^{t-s} \pi_i(\zeta) d\zeta}
$$

Which results in an expected dividend of

$$
\mu_i(s, t) = \int_{\theta=0}^{1} \theta \xi_i(\theta|x, t) d\theta = \frac{\int_{\theta=0}^{1} \theta^{s+1} (1 - \theta)^{t-s} \pi_i(\theta) d\theta}{\int_{\theta=0}^{1} \theta^s (1 - \theta)^{t-s} \pi_i(\theta) d\theta}
$$

For all $\theta \in [0, 1]$ and $i \in \mathcal{I}$, it can be shown that $\lim_{t \to \infty} \mu_i(s, t) = \theta$
Assume there is only one type and that we have seen $s$ payments in $t$ periods.

At what price would the asset trade?

$$p_i^*(s, t) = \sum_{\tau=1}^{\infty} \left( \frac{1}{1 + r} \right)^\tau \mu_i(s, t) = \frac{1}{r} \mu_i(s, t)$$

From this point forward, I will refer to $\mu_i(s, t)$ as type i’s *fundamental value.*
Two definitions of optimist

- **Definition 1**: Trader $k$ is a *global optimist* if $\mu_k(s, t) \geq \mu_i(s, t)$ for all $i \in \mathcal{I}$ and all histories $(s, t)$.

- **Definition 2**: Trader $k$ is a *local optimist* if there exists a history $(s, t)$ such that for all histories $(s', t')$ following $(s, t)$, $\mu_k(s', t') \geq \mu_i(s', t')$.

Why do we care about optimists?
What does an optimist look like?
We would like to find conditions on the priors which ensure no optimists (local or global) exist.

- **Definition 3**: Beliefs $\{\pi_i\}_{i \in I}$ satisfy *perpetual switching* if, for every $i \in I$ and history $(s, t)$, there exists $j \neq i$ and a history $(s', t')$ following $(s, t)$ such that $\mu_j(s', t') > \mu_i(s', t')$. 
Different concept to think about what is required to get exclude optimists

- **Definition 4:** Trader \( k \) is *rate dominant* if
  \[
  \frac{d}{d\theta} \ln(\pi_k(\theta)) \geq \frac{d}{d\theta} \ln(\pi_i(\theta))
  \]
  for all \( i \in \mathcal{I} \) and \( \theta \in [0, 1] \)

- **Note:** Trader \( k \) is rate dominant if and only if his prior density satisfies the monotonic likelihood ratio property with all other priors

Can be shown that

- global optimist \( \Leftrightarrow \) local optimist \( \Leftrightarrow \) rate dominant
A simple class of priors that are clean in seeing the equivalence

Rate dominance is then:

\[
d/d\theta \ln(\pi_i(\theta)) = \frac{\alpha_i - 1}{\theta} - \frac{\beta_i - 1}{1 - \theta}
\]

The difference can be written

\[
d/d\theta \ln(\pi_k(\theta)) - d/d\theta \ln(\pi_i(\theta)) = \frac{\alpha_k - \alpha_i}{\theta} + \frac{\beta_i - \beta_k}{1 - \theta}
\]

Thus rate dominant iff \( \alpha_k \geq \alpha_i \) and \( \beta_k \leq \beta_i \)
Global optimist is $\mu_k(s, t) > \mu_i(s, t)$

The beta distribution is a conjugate prior for a Bernoulli likelihood, so:

- If there are $s$ realizations of the dividend in $t$ periods then posterior is a beta distribution with parameters $(\alpha = \alpha_i + s, \beta = \beta_i + t - s)$
- Mean of a beta distribution is just $\frac{\alpha}{\alpha + \beta}$

Thus global optimist is just

$$\frac{\alpha_k + s}{\alpha_k + \beta_k + t} \geq \frac{\alpha_i + s}{\alpha_i + \beta_i + t} \iff \alpha_k \geq \alpha_i \& \beta_k \leq \beta_i$$
Recall the first slide. Prices will satisfy

\[ p_t = \frac{1}{1 + r} E_t[y_{t+1} + p_{t+1}] \]

What will the price be that they can get tomorrow? It will be the price at which the most optimistic agent tomorrow values it.
Define

$$\mu^*(s, t) = \max_{i \in I} \mu_i(s, t)$$

Then prices satisfy

$$P(s, t) = \frac{1}{1 + r} [\mu^*(s, t)(1 + P(s + 1, t + 1)) + (1 - \mu^*(s, t))P(s, t + 1)]$$

or (in terms of the risk-free asset)

$$p(s, t) = \frac{1}{1 + r} [\mu^*(s, t)(r + p(s + 1, t + 1)) + (1 - \mu^*(s, t))p(s, t + 1)]$$
Theorem 2:

• If trader $k$ is an optimist, then $p(s, t) = \mu_k(s, t)$ for all histories $(s, t)$.

• If there is no optimist, then $p(s, t) > \mu_i(s, t)$ for all histories $(s, t)$ and $i \in \mathcal{I}$.

• As $t \to \infty$, $p(\theta t, t) \to \mu_i(\theta t, t) \to \theta$ for all $\theta \in [0, 1]$ and traders $i \in \mathcal{I}$.
ECONOMY WITH PERPETUAL SWITCHING

No Optimist

Prior Beliefs

Agent 1
Agent 2

Evolution of Means

Prices

Percent Difference in \( p(s, t) \) and \( \mu^*(s, t) \)
Agent 1 is Optimist

Prior Beliefs

Evolution of Means

Prices

Percent Difference in $p(s, t)$ and $\mu^*(s, t)$