Dynamic Competitive Economies with Complete Markets and Collateral Constraints

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Motivation

1. Simplest exchange economy
2. Complete mkts (i.e. a complete set of Arrow securities)
3. With collateral constraints (i.e. short positions need to be backed up)
   - Equivalence between time-0 trading (commodity mkts) and sequential trading (financial mkts) eqlm.?
   - The role of collateral constraints for welfare?
The physical economy

- An infinite horizon stochastic exchange economy w/ a single perishable good
- Uncertainty: an event tree w/ node $\sigma = s^t$
- $S$ exogenous shocks, Markov transition matrix $\pi(s, s')$
- $H$ infinitely lived agents w/ time-separable discounted utility
- Two-part endowments for agent $h$:
  1. $e^h(s_t)$
  2. A given share $\theta^h(s^{-1}) \geq 0$ of a Lucas tree at time 0 (unit net supply), w/ $d(s_t)$ and tradable at price $q(\sigma)$

$$\omega^h(s_t) = e^h(s_t) + \theta^h(s^{-1})d(s_t)$$
Time-0 trading w/ limited pledgeability

- Key assumption: $e^h(s_t)$ cannot be sold in advance
- **Def.**: Consumption ($c^h(\sigma)$) and prices ($\rho(\sigma)$) s.t.
  1. Mkt clearing: $\sum_{h \in H} (c^h(\sigma) - \omega^h(\sigma)) = 0, \forall \sigma$
  2. Optimizing:

$$\forall h, (c^h(\sigma)) \in \arg \max_{c \geq 0} U^h(c)$$

s.t.

$$\sum_{\sigma} \rho(\sigma)c(\sigma) \leq \sum_{\sigma} \rho(\sigma)\omega^h(\sigma)$$

$$\sum_{\sigma \geq s^t} \rho(\sigma)c(\sigma) \geq \sum_{\sigma \geq s^t} \rho(\sigma)e^h(\sigma), \forall s^t$$

- Ex. At node $s^t$, $e^h(s_{t+1})$ cannot be sold at $s^t$, reserved for consumption afterwards. Can only be traded upon $s_{t+1}$.
- Binding when $e^h(\sigma)$ is large relative to dividend.
Sequential trading w/ collateral constraints

1. Long in shares of the tree $\theta(s^t)$ at price $q(s^t)$
2. Long/short in one-period securities (payoff $b_j(s_{t+1}) \geq 0$) $\phi_j^+(s^t)$ and $\phi_j^-(s^t)$ at prices $p_j(s^t)$
   - Key assumption: can default at no cost $\rightarrow$ for each unit of security $j$ sold short, need to hold $k_{j+1}^j$ units of the tree and $k_i^j$ units of each security $i$
   - Endogenous payoff and endogenous constraints

$$f_j(s^{t+1}) = \min\{b_{j}(s_{t+1}), \sum_{i=1}^{J} k_{j}^{i} f_{i}(s^{t+1}) + k_{j+1}^{j}(q(s^{t+1}) + d(s_{t+1}))\}$$

- Assume the tree is used as a collateral for each security $j$, directly or indirectly.
Sequential trading with collateral constraints

1. Long in shares of the tree $\theta(s^t)$ at price $q(s^t)$
2. Long/short in one-period securities (payoff $b_j(s_{t+1}) \geq 0$) $\phi_{j+}(s^t)$ and $\phi_{j-}(s^t)$ at prices $p_j(s^t)$
   - Key assumption: can default at no cost $\rightarrow$ for each unit of security $j$ sold short, need to hold $k_{j+1}^j$ units of the tree and $k_i^j$ units of each security $i$
   - Endogenous payoff and endogenous constraints

$$f_j(s^{t+1}) = \min\{b_j(s_{t+1}), \sum_{i=1}^J k_i^j f_i(s^{t+1}) + k_{j+1}^j(q(s^{t+1}) + d(s_{t+1}))\}$$

- Assume the tree is used as a collateral for each security $j$, directly or indirectly.
Sequential trading w/ collateral constraints

- **Def.** Choices \((c^h(\sigma), \theta^h(\sigma), ((\phi^h_+(\sigma), \phi^h_-(\sigma))))\), prices \((p(\sigma), q(\sigma))\), and payoffs \(f(\sigma)\) s.t.

1. Mkt clearing
2. Optimizing:

\[
\forall h, \text{Choices} \in \arg \max_{\theta \geq 0, \phi_+ \geq 0, \phi_- \leq 0, c \geq 0} U^h(c) \\
\text{s.t.} \\
\begin{align*}
c(s^t) &= e^h(s_t) + \phi(s^{t-1}) \cdot f(s^t) + \theta(s^{t-1})(q(s^t) + d(s_t)) \\
&\quad - \theta(s^t)q(s^t) - \phi(s^t) \cdot p(s^t), \forall s^t \\
\theta(s^t) + \sum_j k^j_{j+1}(s^t)\phi^j_-(s^t) &\geq 0, \forall s^t \\
\phi^j_+(s^t) + \sum_i k^i_j(s^t)\phi^i_-(s^t) &\geq 0, \forall s^t, \forall j
\end{align*}
\]
Equivalence: an example

- Three agents, two periods, three states in 2nd period
- Tree pays 1 unit in each state, every agent holds initially 4 trees
- Non-pledgeable endowment in 2nd period:
  \[ e^1 = (0, 6, 9), \quad e^2 = (6, 9, 0), \quad e^3 = (9, 0, 6) \]
- Time-0 trading eqlm. w/ limited pledgeability:
  \[ c^1 = c^2 = c^3 = (9, 9, 9) \]
- With only trees as collateral, a complete set of Arrow securities cannot complete the mkt: agent 1 holds 4 trees, \((+5, -1, -4)\) Arrow securities for 3 states → violating collateral constraints
Equivalence: an example

- With both trees and financial assets as collateral: “pyramiding” (Geanakoplos and Zame (2002, 2009))
  \[ b^1 = (0, 1, 1), \quad b^2 = (0, 0, 1) \]

- Tree for security 1, security 1 for security 2
  1. \( \theta^1 = 9, \phi^{1-}_1 = -9, \phi^{1+}_1 = 3, \phi^{1-}_2 = -3 \)
  2. \( \theta^2 = 3, \phi^{2-}_1 = -3, \phi^{2+}_2 = 9 \)
  3. \( \theta^3 = 0, \phi^{3+}_1 = 9, \phi^{3-}_2 = -6 \)

- Collateral constraints constrain how trading is conducted, pyramiding (in this case) “completes” the mkt through “bypassing” these constraints.
Equivalence: an example

- With both trees and financial assets as collateral: “pyramiding” (Geanakoplos and Zame (2002, 2009))
  \[ b^1 = (0, 1, 1), b^2 = (0, 0, 1) \]

- Tree for security 1, security 1 for security 2
  1. \( \theta^1 = 9, \phi_{1-}^1 = -9; \phi_{1+}^1 = 3, \phi_{2-}^1 = -3 \)
  2. \( \theta^2 = 3, \phi_{1-}^2 = -3, \phi_{2+}^2 = 9 \)
  3. \( \theta^3 = 0, \phi_{1+}^3 = 9, \phi_{2-}^3 = -6 \)

- Collateral constraints constrain how trading is conducted, pyramiding (in this case) “completes” the mkt through “bypassing” these constraints.
Equivalence: intuition for proof

- “Re-write” the time-0 trading eqlm. w/ limited pledgeability as an eqlm w/ intermediaries
- Intermediaries buy trees from agents and issue tree options (option $j = s$ delivers one tree iff shock $s$ realizes), and have zero profit
- Only long positions in tree options
**Equivalence: intuition for proof**

- “Re-write” the time-0 trading eqlm. w/ limited pledgeability as an eqlm w/ intermediaries
- Intermediaries buy trees from agents and issue tree options (option \( j = s \) delivers one tree iff shock \( s \) realizes), and have zero profit
- Only long positions in tree options
- Construct a rich enough asset structure to replicate pay-offs achieved with tree options (thus portfolio holdings, prices, and consumption allocations), s.t. collateral constraints
  - \( S - 1 \) securities:
    - \((0, 1, 1, \ldots, 1)\), \((0, 0, 1, \ldots, 1)\), \ldots, \((0, \ldots, 0, 1, 1)\), \((0, \ldots, 0, 1)\)
  - Tree for security 1, security 1 for security 2, \ldots, security \( S - 2 \) for security \( S - 1 \)
Pareto efficiency

- Necessary and sufficient condition: collateral (dividends) is “sufficiently large,” or, non-pledgeable endowment is relatively small
- Limited pledgeability constraints in stationary environment: with \( \rho(s^t) = u^h'(c^h(s_t), s_t)\beta^t\pi(s^t) \),

\[
E \left( \sum_{t=1}^{\infty} \beta^t u^h'(c^h(s_t), s_t)(c^h(s_t) - e^h(s)) \right) \geq 0
\]

- With no aggregate uncertainty or identical CRRA preferences, can derive simpler constraints.
• When collateral is small, construct a constrained inefficient example.
• Discuss stationarity: sufficient conditions the existence of Markov eqlm.