Dynamic Competitive Economies with Complete Markets and Collateral Constraints

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Motivation

1. Simplest exchange economy
2. Complete mkts (i.e. a complete set of Arrow securities)
3. With collateral constraints (i.e. short positions need to be backed up)

- Equivalence between time-0 trading (commodity mkts) and sequential trading (financial mkts) eqlm.?
- The role of collateral constraints for welfare?

The physical economy

- An infinite horizon stochastic exchange economy w/ a single perishable good
- Uncertainty: an event tree w/ node $\sigma = s^t$
- S exogenous shocks, Markov transition matrix $\pi(s, s')$
- H infinitely lived agents w/ time-separable discounted utility
- Two-part endowments for agent h:

1. $e^h(s_t)$
2. A given share $\theta^h(s^{-1}) \geq 0$ of a Lucas tree at time 0 (unit net supply), w/ $d(s_t)$ and tradable at price $q(\sigma)$

$$\omega^h(s_t) = e^h(s_t) + \theta^h(s^{-1})d(s_t)$$

Time-0 trading w/ limited pledgeability

- Key assumption: $e^h(s_t)$ cannot be sold in advance
- **Def.**: Consumption $(c^h(\sigma))$ and prices $(\rho(\sigma))$ s.t.

1. Mkt clearing: $\sum_{h\in\mathcal{H}}(c^h(\sigma)-\omega^h(\sigma))=0, \forall\sigma$
2. Optimizing:

$$\begin{aligned}
\forall h, (c^h(\sigma)) &\in \arg\max_{c\geq 0} U^h(c) \\
&s.t. \\
\sum_{\sigma} \rho(\sigma)c(\sigma) &\leq \sum_{\sigma} \rho(\sigma)\omega^h(\sigma) \\
\sum_{\sigma\succeq s^t} \rho(\sigma)c(\sigma) &\geq \sum_{\sigma\succeq s^t} \rho(\sigma)e^h(\sigma), \forall s^t
\end{aligned}$$

- Ex. At node s^t, $e^h(s_{t+1})$ cannot be sold at s^t, reserved for consumption afterwards. Can only be traded upon s_{t+1}.
- Binding when $e^h(\sigma)$ is *large* relative to dividend.

Sequential trading w/ collateral constraints

1. Long in shares of the tree $\theta(s^t)$ at price $q(s^t)$
2. Long/short in one-period securities (payoff $b_j(s_{t+1}) \geq 0$) $\phi_{j+}(s^t)$ and $\phi_{j-}(s^t)$ at prices $p_j(s^t)$

- Key assumption: can default at no cost $\rightarrow$ for each unit of security j sold short, need to hold k^j_{J+1} units of the tree and k^j_i units of each security i
- Endogenous payoff and endogenous constraints

$$f_j(s^{t+1}) = \min\{b_j(s_{t+1}), \sum_{i=1}^{J} k^j_i f_i(s^{t+1}) + k^j_{J+1}(q(s^{t+1}) + d(s_{t+1}))\}$$

- Assume the tree is used as a collateral for each security j, directly or indirectly.

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Sequential trading w/ collateral constraints

- **Def.** Choices $(c^h(\sigma), \theta^h(\sigma), ((\phi_+^h(\sigma), \phi_-^h(\sigma))),$ prices $(p(\sigma), q(\sigma))$, and payoffs $f(\sigma)$ s.t.

1. Mkt clearing
2. Optimizing:

$$
\begin{aligned}
\forall h, Choices \quad &\in \quad \arg\max_{\theta\geq 0, \phi_+\geq 0, \phi_-\leq 0, c\geq 0} U^h(c) \\
&s.t. \\
c(s^t) \quad &= \quad e^h(s_t) + \phi(s^{t-1}) \cdot f(s^t) + \theta(s^{t-1})(q(s^t) + d(s_t)) \\
&\quad -\theta(s^t)q(s^t) - \phi(s^t) \cdot p(s^t), \forall s^t \\
\theta(s^t) \quad &+ \quad \sum_j k_{J+1}^j(s^t)\phi_{j-}(s^t) \geq 0, \forall s^t \\
\phi_{j+}(s^t) \quad &+ \quad \sum_i k_j^i(s^t)\phi_{i-}(s^t) \geq 0, \forall s^t, \forall j
\end{aligned}
$$

Equivalence: an example

- Three agents, two periods, three states in 2nd period
- Tree pays 1 unit in each state, every agent holds initially 4 trees
- Non-pledgeable endowment in 2nd period:

$$e^1 = (0, 6, 9), e^2 = (6, 9, 0), e^3 = (9, 0, 6)$$

- Time-0 trading eqlm. w/ limited pledgeability: $c^1 = c^2 = c^3 = (9, 9, 9)$
- With only trees as collateral, a complete set of Arrow securities cannot complete the mkt: agent 1 holds 4 trees, $(+5, -1, -4)$ Arrow securities for 3 states $\rightarrow$ violating collateral constraints

Equivalence: an example

- With both trees and financial assets as collateral: "pyramiding" (Geanakoplos and Zame (2002, 2009))

$$b^1 = (0, 1, 1), b^2 = (0, 0, 1)$$

- Tree for security 1, security 1 for security 2

1. $\theta^1 = 9, \phi^1_{1-} = -9; \phi^1_{1+} = 3, \phi^1_{2-} = -3$
2. $\theta^2 = 3, \phi^2_{1-} = -3, \phi^2_{2+} = 9$
3. $\theta^3 = 0, \phi^3_{1+} = 9, \phi^3_{2-} = -6$

- Collateral constraints constrain how trading is conducted, pyramiding (in this case) "completes" the mkt through "bypassing" these constraints.

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Equivalence: intuition for proof

- "Re-write" the time-0 trading eqlm. w/ limited pledgeability as an eqlm w/ intermediaries
- Intermediaries buy trees from agents and issue tree options (option $j = s$ delivers one tree iff shock s realizes), and have zero profit
- Only long positions in tree options

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- "Re-write" the time-0 trading eqlm. w/ limited pledgeability as an eqlm w/ intermediaries
- Intermediaries buy trees from agents and issue tree options (option $j = s$ delivers one tree iff shock s realizes), and have zero profit
- Only long positions in tree options
- Construct a rich enough asset structure to replicate pay-offs achieved with tree options (thus portfolio holdings, prices, and consumption allocations), s.t. collateral constraints
  - $S - 1$ securities: $(0, 1, 1, \ldots, 1), (0, 0, 1, \ldots, 1), \ldots, (0, \ldots, 0, 1, 1), (0, \ldots, 0, 1)$
  - Tree for security 1, security 1 for security 2, ..., security $S - 2$ for security $S - 1$

Pareto efficiency

- Necessary and sufficient condition: collateral (dividends) is "sufficiently large," or, non-pledgeable endowment is relatively small
- Limited pledgeability constraints in stationary environment: with $\rho(s^t) = u^{h'}(c^h(s_t), s_t)\beta^t \pi(s^t)$,

$$u^{h'}(c^h(s), s)(c^h(s) - e^h(s)) \quad + \quad \ldots$$
$$\mathbf{E}\left(\sum_{t=1}^{\infty} \beta^t u^{h'}(c^h(s_t), s_t)(c^h(s_t) - e^h(s_t)) \mid s_0 = s\right) \quad \geq \quad 0$$

- With no aggregate uncertainty or identical CRRA preferences, can derive simpler constraints.

Other things

- When collateral is small, construct a constrained inefficient example.
- Discuss stationarity: sufficient conditions the existence of Markov eqlm