Optimal Beliefs, Asset Prices and the Preference for Skewed Returns.

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Motivation

- Some facts: Households are not well diversified / Households tend to hold large amounts of stocks with positive skewness / Positively skewed assets tend to have low exp. returns.

- This paper: Use "Optimal Expectations" (BP,2005) in a portfolio choice problem to address these issues. OE involve 2 stages:

- *Stage 2:* Solve the portfolio problem given *subjective* beliefs.

- *Stage 1:* Choose "optimally" subjective beliefs subject to the *optimal* actions of Stage 2.

Stage 2: *Optimal Portfolio Choice with subjective beliefs*

- 2 period economy, invest in the first period and consume in the second.Initial wealth is unity.

- S states with *subjective* beliefs $\hat{\pi}_s$. Markets are *complete* with price of AD sequrity p_s.

- Portfolio Problem

$$\max_{\{c\}} \hat{E}U(c) = \sum_s \hat{\pi}_s U(c_s)$$

s.t.

$$\sum_s p_s c_s = 1$$

- FOC for the agent

$$\hat{\pi}_s U'(c_s) = \lambda p_s$$

where λ the multiplier on the budget constraint or

$$\frac{U'(c_s)}{U'(c_s)} = \frac{p_s/\hat{\pi}_s}{p_k/\hat{\pi}_k}$$

- Thus if $p_s/\hat{\pi}_s > p_k/\hat{\pi}_k \Rightarrow c_k > c_s$

- The optimal solution is

$$c_s^* = c_s^*(\hat{\pi})$$

- From now on log utility with $c_s = \frac{\hat{\pi}_s}{p_s} = \frac{\hat{\pi}_s}{\pi_s} \cdot \frac{\pi_s}{p_s}$.

Stage 1: *Optimal Beliefs: maximize "Well-Being"*

- Tradeoff beween *benefit* of distorted beliefs (raise *anticipatory* utility) and *cost* of ex-post bad decision making

- Let π denote the *objective* beliefs. Objective: Choose $\hat{\pi} \in \Delta(S)$ to maximize *Well-Being:*

$$\begin{aligned} W &= \frac{1}{2} E[\hat{E}U\left(c^*(\hat{\pi})\right) + U(c^*(\hat{\pi}))] \\ &= \frac{1}{2}[\underbrace{\hat{E}U\left(c^*(\hat{\pi})\right)}_{\text{anticipatory utility of optimism}} + \underbrace{EU(c^*(\hat{\pi}))}_{\text{cost of bad decisions}}] \end{aligned}$$

- Well being is the average felicity under rational expectations.

- In case of log utility W becomes

$$W = \frac{1}{2}[\sum_s \hat{\pi}_s \ln \frac{\hat{\pi}_s}{p_s} + \sum_s \pi_s \ln \frac{\hat{\pi}_s}{p_s}] = \frac{1}{2}[\underbrace{\sum_s \hat{\pi}_s \ln \frac{\hat{\pi}_s}{\pi_s}}_{\text{relative entropy}} + \underbrace{\sum_s \hat{\pi}_s \ln \frac{\pi_s}{p_s}}_{\hat{E}U(c^*(\pi))} + \underbrace{\sum_s \pi_s \ln \frac{\hat{\pi}_s}{p_s}}_{EU(c^*(\hat{\pi}))}]$$

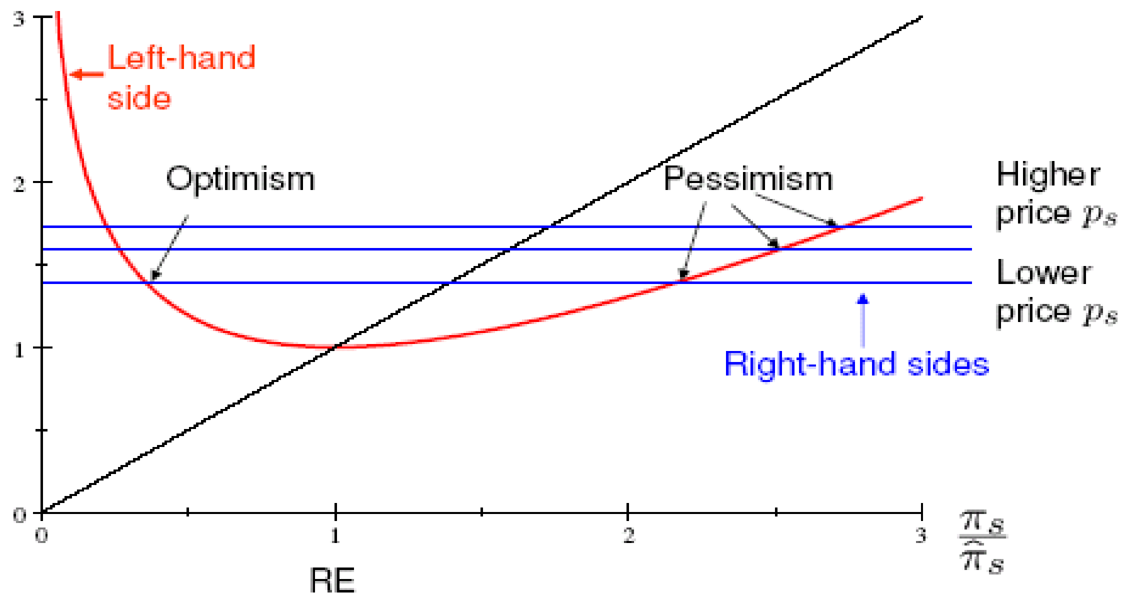
- FOC

$$\frac{\pi_s}{\hat{\pi}_s} - \ln \frac{\pi_s}{\hat{\pi}_s} = \mu - 1 + \ln \frac{p_s}{\pi_s}, \text{ all } s$$

- SOC:

$$\hat{\pi}_s(1 - \frac{\pi_{s'}}{\hat{\pi}_{s'}}) \leq \hat{\pi}_{s'}(\frac{\pi_s}{\hat{\pi}_s} - 1), \forall s \neq s'$$

- From SOC we see that there is at *most one* state that has its probability biased upwards ($\hat{\pi}_{s'} > \pi_{s'}$).



$$\frac{\pi_s}{\hat{\pi}_s} - \ln \frac{\pi_s}{\hat{\pi}_s} = \mu - 1 + \ln \frac{p_s}{\pi_s}, \text{all } s$$

- Non actuarially fair prices imply no RE ($\frac{\pi_s}{\hat{\pi}_s} \neq 1$)

- *Proposition:* RE are optimal only if $S = 2$ *and* $\pi = 1/2$ *and* prices are actuarially fair ($p_1/\pi_1 = p_2/\pi_2$).

- Which state to bias upwards? The "cheapest state" in terms of p_s/π_s and π_s.

  - If states are actuarially fair (p_s/π_s =const $\forall s$) then the investor overestimates the state with the lowest probability.

  - If states are equally likely then the investor bias the one with the lowest p/π

  - The investor biases upwards the state with the smallest p/π and π.

Actuarially Fair prices and Portfolio Choice ($S > 2$)

- If $p_s/\pi_s = 1$, then investor biases the upwards the state with the smallest probability and biases downwards *equally* all the other states.

- He consumes at the state with the *lowest* probability π (preference for skewed asset)

$$\bar{c} = \frac{\hat{\pi}}{\pi} > 1$$

and in *all* other states

$$\underline{c} = \frac{1 - \hat{\pi}}{1 - \pi} < 1$$

with

$$\pi\bar{c} + (1 - \pi)\underline{c} = 1$$

- Thus in contrast to RE, the optimal portfolio is *not* riskfree $c_s = c \forall s$

- Optimal Portfolio (for $\sum p_s = 1$)

$$p\bar{c} + (1 - p)\underline{c}$$

- Portfolio can be written as

$$p(\bar{c} - \underline{c}) + \underline{c}$$

  with interpretation: invest $\underline{c}$ in the risk-free asset and invest $\bar{c} - \underline{c}$ in the most skewed asset (Two-fund separation property)

General Equilibrium with Optimal Expectations

- With No aggregate uncertainty $C_s = C = 1$ and *RE* we would have actuarialy fair prices

$$p_s = \pi_s, c_s^i = c^i = 1 \forall i \in [0, 1]$$

- With Aggregate uncertainty $C_s \neq C_k$ and *RE* we would have

$$p_s = \frac{\frac{\pi_s}{C_s}}{\sum_s \frac{\pi_s}{C_s}}$$

- No *Aggregate uncertainty* and $\pi_s = \pi = \frac{1}{S}$. Then *OEE* has actuarially fair prices $p_s = \pi_s$ and
  - Each agent biases upwards the probability of one state and consumes $\bar{c} > 1$ and in all other states $\underline{c} < 1$.
  - Exactly $\frac{1}{S}$ agents have to bias each state in order to have market clearing. GE creates heterogenous beliefs!
- Generalization with *Aggregate Uncertainty*: Let $\pi_s = \pi$. If C_s is such that $\underline{c} \leq C_s \leq \bar{c}$ with $\sum_s C_s = \pi\bar{c} + (S-1)\underline{c}$ $\exists$ OEE with $p_s = \pi_s = \pi$ and a fraction $\lambda_s = \frac{C_s - \underline{c}}{\bar{c} - \underline{c}}$ that consumes $\bar{c}$.
- Aggregate risk is not priced (holds only for $\pi_s = \pi$)! With π_s not equal you can generate also assets with positive skewness that have lower expected returns.