Competitive Equilibria of Economies with a Continuum of Consumers and Aggregate Shocks

Jianjun Miao

Forthcoming, JET

Motivation

- Does there exist any sequential competitive equilibrium (SCE) for Bewley-type model with aggregate productivity shocks?
- If so, can we find a recursive characterization of the SCE?

Model (I)

- Time is discrete: t = 0,1,2,…
- Probability space: $(\Omega \times \mathbf{Z}^{\infty}, F, \mathrm{P})$

  (a) individual shock: $w \in \Omega$

  (b) aggregate shock: $z^{\infty} = (z_0, z_1, z_2, \ldots) \in \mathbf{Z}^{\infty}$
- Consumers are distributed on I=[0,1], according to the Lebesgue measure f

  (a) ex ante identical: same preference, endowment shocks draw from the same distribution.

  (b) ex post heterogeneous: idiosyncratic endowment shocks.
- Labor endowment process: $(s_t^i)_{t \geq 0},\ s_t^i : \Omega \times \mathbf{Z}^{\infty} \rightarrow \mathbf{S} \subset R_+,\ s_0^i$ constant
- Information structure: $\mathrm{F}_t^{\,i} = s\ (\{s_n^i, z_n\}_{n=0}^t)$
- Asset process: $(a_t^i)_{t \geq 0},\ a_t^i : \Omega \times \mathbf{Z}^t \rightarrow \mathbf{A} \subset R,\ a_0^i$ constant, a_t^i is $\mathrm{F}_t^{\,i}$-measurable
- Initial distribution of asset holdings and endowment shocks:

$$l_0(A \times S) = f(i \in I:\ (a_0^i, s_0^i) \in A \times S),\ \ A \times S \in \mathrm{B}(\mathbf{A}) \times \mathrm{B}(\mathbf{S})$$

Model (II)

- Budget constraint: $c_t^i + a_{t+1}^i = (1+r_t)\cdot a_t^i + w_t \cdot s_t^i$
- No borrowing constraint: $a_t^i \geq 0, \ \forall i \in I, \quad i.e. \ \mathbf{A} = [0, \infty)$
- Assumption 1: $\mathbf{Z} \subset [\underline{z}, \overline{z}] \subset R_{++}$, countable set with discrete topology

  $\mathbf{S} \subset R_+$, compact
- Assumption 2: for $f - i.e., \ \forall i$

(a) given the history $(s^{it}, z^t) = (s^t, z^t)$

(s_{t+1}^i, z_{t+1}) is draw from the distribution $Q_{t+1}(\cdot, s^t, z^t)$

(b) $Q_{t+1}(S \times Z, \cdot)$ is measurable for $\forall S \times Z \in \mathrm{B}(\mathbf{S}) \times \mathrm{B}(\mathbf{Z})$

(c) Q_{t+1} has the Feller property:

$\int h(s', z') Q_{t+1}(ds', dz', \cdot)$ is a continuous function on $\mathbf{S}^t \times \mathbf{Z}^t$

for all $h \in C(\mathbf{S} \times \mathbf{Z})$

Model (III)

- Preferences: $U(c^i) = E[\sum_{t=0}^{\infty} b^t u(c_t^i)],\ c_t^i \in C^i,\ 0 < b < 1$
- Assumption 3: $u : R_+ \to R$ is bounded, continuous and strictly concave.
- Assumption 4: for each *t*, $s_t^i : I \times \Omega \times \mathbf{Z}^\infty \to \mathbf{S}$ is $\mathrm{B}(I) \times \mathrm{F}_t$ – measurable
- Admissibility:

  An allocation $\{(c_t^i, a_{t+1}^i)_{t \geq 0}\}_{i \in I}$ is admissible if both $c_t^i = c_t(i, w, z^t)$ and $a_{t+1}^i = a_{t+1}(i, w, z^t)$ are $\mathrm{B}(I) \times \mathrm{F}_t$ – measurable

Remark:

1. $\mathrm{F}_t = \vee_{i \in I} \mathrm{F}_t^i$ is the smallest s -algebra containing F_t^i for all i
2. Admissibility is guaranteed by assumption 4. (see Karatzas,et al,1994)
3. Consumer behavior: $\sup_{\{(c_t^i, a_{t+1}^i)_{t \geq 0}\}} u(c^i)$

   s.t. budget constraint,

   borrowing constraint for all *t*

Model (IV)

- Single firm production: $Y_t = z_t F(K_t, N_t) + (1-d)K_t$

  where K_t is $\mathcal{F}_{t-1}$ – measurable, and N_t is $\mathcal{F}_t$ – measurable.
- Assumption 5:

  (a) $0 \le N_t \le \hat{N}$

  (b) $F : R_+ \times R_+ \to R_+$ is homogeneous of degree one, strictly increasing, stictly concave, continuously differentiable.

  $$F(0,.) = F(.,0) = 0, \lim_{K \to 0} F_1(K, \hat{N}) = \infty, \lim_{K \to \infty} F_1(K, \hat{N}) = 0$$

Remark:

1. maximum sustainable $\hat{K}$ is determined by $\bar{z}F(K, \hat{N}) = dK$
2. Profit maximization (period by period):

$$\max_{\{K_t, N_t\}} Y_t - (1+r_t)K_t - w_t N_t$$

$$f.o.c. \quad r_t = z_t F_1(K_t, N_t) - d$$

$$w_t = z_t F_2(K_t, N_t), \text{ both are } \mathcal{F}_t - \text{measurable}$$

Definition of Sequential Competitive Equilibrium

- A ***sequential competitive equilibrium*** $\{[(c_t^i, a_{t+1}^i)_{t\geq 0}]_{i\in I}, (r_t, w_t)_{t\geq 0}\}$ consists of an ***admissible allocation*** $[(c_t^i, a_{t+1}^i)_{t\geq 0}]_{i\in I}$ and ***price process*** $(r_t, w_t)_{t\geq 0}$ such that:

  (i) Given prices, the allocation solves consumer's utility maximization problem for $f - i.e.\ \forall i$

  (ii) Given prices, the firm maximizes profits so that the *f.o.c.*s are satisfied.

  (iii) Markets clear, i.e., for all $t \geq 0$

  $$\int_I s_t^i f(di) = N_t, \ \int_I a_t^i f(di) = K_t,$$

  $$C_t + K_{t+1} = z_t F(K_t, N_t) + (1-d)K_t, \ \ where\ C_t = \int_I c_t^i f(di)$$

Remark: by Walras law, one of the above three markets clear condition is redundant.

Introducing Aggregate Distribution

- Individual state: a pair of individual asset holdings and the history of individual shocks, (a_t^i, s^{ti})
- Aggregate distribution: $l_t \in \mathrm{P}(\mathbf{A} \times \mathbf{S}^t)$

$$l_t(A \times B) = f(i \in I : (a_t^i, s^{ti}) \in A \times B), \ \ where\ A \times B \in \mathrm{B}(\mathbf{A}) \times \mathrm{B}(\mathbf{S}^t)$$

Remark:

(a) l_t is a random measure since $a_t^i = a_t^i(w, z^{t-1})$ and $s_t^i = s_t^i(w, z^t)$ are random variables.

(b)
$$K_t = \int_I a_t^i f(di) = \int_{\mathbf{A}\times\mathbf{S}^t} a\, l_t(da, ds^t),$$
$$N_t = \int_I s_t^i f(di) = \int_{\mathbf{A}\times\mathbf{S}^t} s\, l_t(da, ds^t),$$
$$C_t = \int_I c_t^i f(di) = (1 + r_t)K_t + w_t N_t - K_{t+1}$$

(c) The *f.o.c.s* for firm and (b) induce the following pricing functions:
$$r_t(l_t, z_t) = z_t F_1\left(\int_{\mathbf{A}\times\mathbf{S}^t} a\, l_t(da, ds^t), \int_{\mathbf{A}\times\mathbf{S}^t} s\, l_t(da, ds^t)\right) - d$$
$$w_t(l_t, z_t) = z_t F_2\left(\int_{\mathbf{A}\times\mathbf{S}^t} a\, l_t(da, ds^t), \int_{\mathbf{A}\times\mathbf{S}^t} s\, l_t(da, ds^t)\right)$$

One-person Decision Problem

- Assume $l_t : \mathbf{Z}^t \to P(\mathbf{A} \times \mathbf{S}^t)$ does not depend on $w \in \Omega$
  Note: this will be guaranteed by "*conditional no aggregate uncertainty*".
- Let $m = \{l_t\}_{t \geq 0} \in P_\infty(\mathbf{A} \times \mathbf{S}) = \times_{t=0}^{\infty} P(\mathbf{A} \times \mathbf{S}^t)^{\mathbf{Z}^t}$
- For given m, let $V_t(a_t, s^t, z^t, m)$ be the value function of the consumer at t. Then

$$V_t(a_t, s^t, z^t, m) = \sup_{\{a_{t+1}\}} u((1 + r_t(l_t(z_t), z_t))a_t + w_t(l_t(z_t), z_t)s_t - a_{t+1})$$

$$+ b \int_{\mathbf{S} \times \mathbf{Z}} V_{t+1}(a_{t+1}, s^{t+1}, z^{t+1}, m) Q_{t+1}(ds_{t+1}, dz_{t+1}, s^t, z^t),$$

$$where\ a_{t+1} \in \Gamma(a_t, s_t, z_t, l_t(z_t)) = [0, (1 + r_t(l_t(z_t), z_t))a_t + w_t(l_t(z_t), z_t)s_t]$$

- Lemma: Given assumption 1-5, there is a unique sequence of value functions and a unique sequence of continuous policy functions that solves the one-person decision problem.

Conditional No Aggregate Uncertainty

- No aggregate uncertainty: for $X = (X_t)_{t\geq 0}, X_t : I \times \Omega \to \mathbf{D}$

  $\exists$ *nonrandom measure* v,

  $s.t., f(i \in I : X(i,w) \in D) = v(D), D \in \mathrm{B}(\mathbf{D}^{\infty}) \quad for\ \mathrm{P}-a.e.\ w$
- Conditional no aggregate uncertainty:

  $for\ X = (X_t)_{t\geq 0}, X_t : I \times \Omega \times \mathbf{Z}^{\infty} \to \mathbf{D} \quad , given\ z^{\infty} \in \mathbf{Z}^{\infty}$

  X satisfies no aggregate uncertainty condition

- Assumption 6. s^i satisfies conditional no aggregate uncertainty, relative to the probability space $(\Omega \times \mathbf{Z}^{\infty}, \mathrm{F}, \mathrm{P})$
- Lemma 2. Under assumption 6, given z^{∞} , l_t evolves according to

$$l_{t+1}(z^{t+1}(A\times B)) = \int_{A\times B_1} 1_A(g_{t+1}(a_t, s^t, z^t, m))Q_{t+1}(B_2, z_{t+1}, s^t, z^t)\, l_t(da_t, ds^t)(z^t),$$

where $A \in \mathrm{B}(\mathbf{A}), B = B_1 \times B_2 \in \mathrm{B}(\mathbf{S}^t) \times \mathrm{B}(\mathbf{S})$

Note that l_t does not depend on w .

Result 1

- Theorem 1. Given assumptions 1-6, there exists a sequential competitive equilibrium. Moreover, the set of equilibrium sequences of aggregate distributions are compact.

Stationarity Assumptions

- Assumption 7. $Q_{t+1}(S\times Z, s^t, z^t) = Q(S\times Z, s_t, z_t)$ *for all* $t, S\times Z \in \mathrm{B}(\mathbf{A})\times \mathrm{B}(\mathbf{Z})$

- Assumption 8. Aggregate labor endowments at any date t is given by a measurable function $N : \mathbf{Z}^t \to (0, \hat{N}]$.

- Markovian property implied by the above stationarity assumptions: past history of individual shocks does not affect current decisions.
- New definition for aggregate distribution:

$$l_t(A\times B) = f(i\in I : (a_t^i, s_t^i) \in A\times B),\ \ A\times B \in \mathrm{B}(\mathbf{A})\times \mathrm{B}(\mathbf{S})$$

Recursive Competitive Equilibrium (RCE)

- Definition: A RCE consists of a measurable policy function *f*, a measurable map T^v , a measurable map *G*, and a measurable pricing functions *r* and *w* such that:

(a) Given the pricing functions, the policy function solves the following problem: $v(a,s,z,l) = \sup_{\{a'\in\Gamma(a,s,z,l)\}} u((1+r(l,z))a + w(l,z)s - a') + b\int_{\mathbf{S}\times\mathbf{Z}} v'(a',s',z',l')Q(ds',dz',s,z)$

$$where\ v' = T^v(z,l,v)(\cdot),\quad l' = G(z,l,v,z'),$$

$$\Gamma(a,s,z,l) = [0,(1+r(l,z))a + w(l,z)s]$$

(b) The firm maximizes profits.

(c) The sequence of aggregate distributions induced by *G* is such that labor market clear:

$$\int_{\mathbf{A}\times\mathbf{S}} s\, l_t(da,ds) = N(z^t),\ \ where\ \ l_{t+1} = G(z_t, l_t, v_t, z_{t+1}),\quad l_0 given$$

(d) The law of motion for aggregate distribution *G* is generated by the individual optimal policy *f*,

$$G(z,l,v,z')(A\times B) = \int_{\mathbf{A}\times\mathbf{S}} 1_A(f(a,s,z,l,v))Q(B,z',s,z)\, l(da,ds)$$

Result 2

- Theorem 2. Given assumptions 1-8, given the initial state $((a_0^i, s_0^i)_{i\in I}, z_0, l_0, v_0)$ a RCE $((f, T^v, G), r, w)$ generates a SCE $\{((a_{t+1}^i, c_t^i)_{t\geq 0})_{i\in I}, (r_t, w_t)_{t\geq 0}$ in which consumer i's expected discounted utility is given by $v_0(a_0^i, s_0^i, z_0, l_0)$

- Theorem 3. Under assumptions 1-8, there exists a RCE. Moreover, for any SCE with the sequence of aggregate distributions $m*$, there exists a payoff equivalent RCE.