Asset Prices in a Heterogeneous-Agent Economy with Portfolio Constraints

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The Contribution of this Paper

- The study the effects of heterogeneity in risk aversion and of leverage constraints on assets behavior.
- To study this economy by an analytical approximation.

The Economy

- Preferences:

$$E_0 \left[\int_0^\infty e^{-\rho t} \frac{1}{\gamma} (C_t^\gamma - 1) dt \right]$$

- Endowment process

$$de_t = \mu_e e_t dt + \sigma_e e_t dZ_{et}$$

Assets

- Short term risk-free bond available in zero supply, paying r_t
- Stock paying the aggregate endowment.

$$\frac{dP_t}{P_t} = \mu_{Pt} dt + \sigma_{Pt} dZ_{Pt}$$

$$\phi_t = \frac{\mu_{Pt} - r_t}{\sigma_{Pt}^2}$$

State of the economy

- $\gamma \equiv \epsilon a$ (so $rra = 1 - \epsilon a$)
- The state is $X = \{W(a)\}$
- At the equilibrium:

$$dX_t = \mu_X(X_t)dt + \sigma_X(X_t)dZ_{Xt}$$

The Seq.tial Problem of the "a"-Agent

$$
\begin{aligned}
J(W(a)_t, X_t) &= \max_{\{C(a)_s, \pi(a)_s\}} E_t \left[\int_t^{\infty} e^{-\rho(s-t)} \frac{C(a)_s^{\epsilon a} - 1}{\epsilon a} ds \right] \\
&\quad s.t. \\
dW(a)_t &= [(r_t + \pi(a)_t(\mu_{Pt} - r_t))W(a)_t - C(a)_t]dt \\
&\quad + \pi(a)_t \sigma_{Pt} W(a)_t dZ_{Pt} \\
dX_t &= \mu_X(X_t)dt + \sigma_X(X_t)dZ_{Xt} \\
\frac{dP_t}{P_t} &= \mu_{Pt}dt + \sigma_{Pt}dZ_{Pt} \\
\pi(a)_t &\leq \overline{\pi} = 1 + \epsilon L
\end{aligned}
$$

Equilibrium

- $\{\pi(a)_t, C(a)_t, P_t, r_t\}$:
  1) Given prices, agents optimize;
  2) Stock market and Goods market clear, $\forall t, \forall X_t$:

$$\sum_a \pi(a)_t \frac{W(a)_t}{\sum_a W(a)_t} = 1$$

$$\sum_a C(a)_t = e_t$$

Useful definitions

$$w(a)_t \equiv \frac{W(a)_t}{\sum_a W(a)_t}$$

$$\overline{a}_t \equiv \sum_a a w(a)_t$$

$$\sigma(a)_t \equiv \sum_a (a^2 - \overline{a}_t) w(a)_t$$

Asymptotic Expansion

- Find the closed form solution of this problem for a particular case (in this case $\epsilon = 0 \Leftrightarrow \gamma = 0$, log-utility)
- Expand the objective function by a power series in the parameter of interest (in this case $\epsilon^0, \epsilon, \epsilon^2, ...$)

Results without constraints ($L = +\infty$)

- Price-Dividend for $\epsilon \to 0$:

$$\frac{P_t}{e_t} = \frac{1}{\rho}\left(1 + \epsilon\overline{a}_t\frac{1}{\rho}\left(\mu_e - \frac{\sigma_e^2}{2}\right)\right) + O(\epsilon^2);$$

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- Mean Returns for $\epsilon \to 0$:

$$E[R_t] = (\mu_e + \rho) - \epsilon \overline{a}_t \left(\mu_e - \frac{\sigma_e^2}{2}\right) + O(\epsilon^2);$$

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- Volatility of the Returns for $\epsilon \to 0$:

$$\sigma(R_t) = \sigma_e + \epsilon^2 \sigma(a)_t \frac{1}{\rho}\left(\mu_e - \frac{\sigma_e^2}{2}\right)\sigma_e + O(\epsilon^3);$$

Results without constraints ($L = +\infty$)

- Portfolio allocation for $\epsilon \to 0$:

$$\pi_t = 1 + \epsilon(a - \overline{a}_t) + O(\epsilon^2)$$

- Wealth distribution allocation for $\epsilon \to 0$:

$$\frac{dw(a)_t}{w(a)_t} = \left(\mu_e - \frac{\sigma_e^2}{2}\right)\epsilon(a - \overline{a}_t)dt + \sigma_e\epsilon(a - \overline{a}_t)dZ_t + O(\epsilon^2)$$

- $e_t \uparrow \Rightarrow w(a^{high}) \uparrow \Rightarrow \overline{a}_t \uparrow$ (pro-cyclical stock-mkt)

Results with constraints

- The Borrowing constraint reduces the variability of the cross-sectional wealth distribution.
- Returns are less volatile.
- The risk free rate is lower, the equity premium bigger.

Spirit of the approximation: case 1

- Consider the following recursive problem:

$$V(k) = \max_{[0\ k^\alpha]} \{\log(k^\alpha - k') + \beta V(k')\}$$

- The close-form solution is:

$$\begin{aligned} V(k) &= \omega + \frac{\alpha}{1-\beta\alpha}\log(k) \\ &= \lim_{\Psi \to 1} \frac{\left(e^{\omega + \frac{\alpha}{1-\beta\alpha}\log(k)} - 1\right)^{\left(1-\frac{1}{\Psi}\right)}}{1-\frac{1}{\Psi}} \end{aligned}$$

Spirit of the approximation: case 1

- Look for the policies functions using:

$$V(k,\Psi) \approx \frac{\left(e^{\omega+\frac{\alpha}{1-\beta\alpha}\log(k)}-1\right)^{(1-\frac{1}{\Psi})}}{1-\frac{1}{\Psi}}$$

OR

$$V(k,\Psi) = \omega+\frac{\alpha}{1-\beta\alpha}\log(k)+(\Psi-1)\frac{\partial V}{\partial \Psi}+\ldots + \frac{(\Psi-1)^2}{2}\frac{\partial^2 V}{\partial \Psi^2}+\ldots+o(\Psi^n)$$

Spirit of the approximation: case 2

- Hansen-Heaton-Li economy:

$$v_t(\Psi) \equiv \frac{V(\Psi)_t}{C_t} = \left[1 + \beta E_t \left[\left(e^{\Delta c_{t+1}} v_{t+1}\right)^{1-\gamma}\right]^{\frac{1-1/\Psi}{1-\gamma}}\right]^{\frac{1}{1-1/\Psi}}$$

$$\Delta c_{t+1} = F x_{t+1}$$

- When $\Psi = 1$:

$$v_t(1) = e^{A+Bx_t} = \lim_{\Psi \to 1} \left[1 + \beta E_t \left[e^{(1-\gamma)(\Delta c_{t+1} + A + Bx_t)}\right]^{\frac{1-1/\Psi}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\Psi}}}$$