Efficiency, Equilibrium, and Asset Pricing with Risk of Default

Fernando Alvarez and Urban J. Jermann

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Outline of the talk

- Complete markets with risk of default.
- Competitive equilibrium with endogeneous solvency constraints.
- Welfare theorems.
- Extent of risk sharing.
- Asset pricing.

The environment

- Pure exchange economy with I agents and 1 good.
- Endowments follow a Markov process: $e_{i,t}(z_t)$
- Option to default and live in autarky forever.
- Agent's utility:

$$U(c)(z^t) = \sum_{s=t}^{\infty} \sum_{z^s \in Z^s} \delta_{t,s}(z^{s-1}) u(c_s(z^s)) \pi(z^s | z_t)$$

where $\delta_{t,t}(z^{t-1}) = 1$ and $\delta_{t,s}(z^{s-1}) = \delta_{t,s-1}(z^{s-2}) \beta(z_{s-1})$

$$U(c)(z^t) = u(c(z^t)) + \beta(z_t) \sum_{z_{t+1} \in Z} U(c)(z^t, z_{t+1}) \pi(z_{t+1} | z_t)$$

Efficient allocation

- An allocation is feasible if it satisfies:
  - Resource feasibility:

$$\sum_{i=1}^{I} c_{i,t}(z^t) = \sum_{i=1}^{I} e_{i,t}(z_t) \quad \forall t, z^t$$

  - Participation constraints:

$$U(c_i)(z^t) \geq U(e_i)(z_t) \quad \forall t, z^t, i$$

- An allocation is (constrained) efficient if it is not Pareto dominated by other feasible allocations.
- With CRRA, stochastic growth and constant β is equivalent to constant e and state contingent β.

Equilibrium (1)

- $\{c_i, a_i, q\}$ is an equilibrium with solvency constraints $\{B_i\}$ for initial conditions $\{a_{i,0}\}$ if:
  - for each i, $\{c_i, a_i\}$ solves:

$$
\begin{aligned}
J_{i,t}(a, z^t) &= \max_{c,\{a_{z'}\}} u(c) + \beta(z) \sum_{z'} J_{i,t+1}(a_{z'}, (z^t, z'))\pi(z'|z_t) \\
e_{i,t}(z^t) &= -a + \sum_{z'} a_{z'} q_t(z_t, z') + c \\
a_{z'} &\geq B_{i,t+1}(z^t, z') \qquad \forall z'
\end{aligned}
$$

  - markets clear.
- Solvency constraints are not too tight if:

$$J_{i,t+1}(B_{i,t+1}(z^{t+1}), z^{t+1}) = U(e_i)(z^{t+1}) \qquad \forall t, z^{t+1}$$

Equilibrium (2)

- Euler conditions:

$$-u'(c_{i,t}(z^t))q_t(z^t,z') + \beta(z)\pi(z'|z^t)u'(c_{i,t+1}(z^t,z')) \leq 0$$

with equality if $a_{i,t+1} > B_{i,t+1}(z^t,z')$.

- Define Arrow prices:

$$q_t^*(z^t,z') = \max_i \left\{ \beta(z) \frac{u'(c^*_{i,t+1}(z^t,z'))}{u'(c^*_{i,t}(z^t))} \pi(z'|z_t) \right\}$$

Def. The implied interest rates for $\{c_i^*\}$ are high if

$$\sum_{t=0}^{\infty} \sum_{z^t} Q_0^*(z^t|z_0) \left(\sum_i c^*_{i,t}(z^t) \right) < \infty$$

Second Welfare Theorem

Prop. Let $\{c_i\}$ be a (constrained) efficient allocation. Then for any agent j at time t with $U(c_j)(z^t, z_{t+1}) > U(e_j)(z^t, z_{t+1})$:

$$\frac{u'(c_{j,t+1})}{u'(c_{j,t})} = \max_i \frac{u'(c_{i,t+1})}{u'(c_{i,t})}$$

Second Welfare Theorem Any constrained efficient allocation that has high implied interest rates can be decentralized as a competitive equilibrium with solvency constraints that are not too tight.

Prop. Autarky can always be decentralized as an equilibrium with solvency constraints that are not too tight, with $B_{i,t}(z^t) = 0$.

First Welfare Theorem

K-L decentralization: $\max_{\{c_i\}} U(c_i)(z_0)$ s.t.

$$\sum_{t=0}^{\infty} \sum_{z^t} (c_{i,t}(z^t) - e_{i,t}(z_t)) Q_O(z^t | z_0) \leq a_{i,0}$$
$$U(c_i)(z^t) \geq U(e_i)(z_t) \qquad \forall t, z^t$$

Prop. An equilibrium with solvency constraints that are not too tight, with implied high interest rates and such that:

$$|u(c_{i,t}(z_t))| \leq \xi_i u'(c_{i,t}(z_t)) c_{i,t}(z_t) \ \ \forall i, t, z^t$$

can be expressed as a K-L equilibrium.

First Welfare Theorem. Under this conditions, the equilibrium is constrained efficient.

Risk sharing

Autarky is the only feasible allocation in any of the following cases:

- β is sufficiently small,
- risk aversion is sufficiently small uniformly,
- variance of the idiosyncratic shock is sufficiently close to zero,
- transition matrix is sufficiently close to identity.

Prop. If some risk sharing is possible, then the implied interest rates are high. Moreover, the solvency constraints are negative.

Asset pricing

- Solvency constraints with complex securities:

$$\sum_{k' \in K_{t+1}} \left[q_{t+1,k'}(z^t, z') + d_{t+1,k'}(z^t, z') \right] a_{i,t+1,k'}(z^t) \geq B_{i,t+1}(z^t, z')$$

- Prices of securities with nonnegative payoffs are generally higher than any of the agents' valuation.

$$\begin{aligned} MV_{i,t,k'}(z^t) &= \sum_{s=0}^{\infty} \sum_{z^s \succsim z^t} \delta_{t,t+s} d_{t+s,k'}(z^s) \frac{u'(c_{i,t+s}(z^s))}{u'(c_{i,t}(z^t))} \pi(z^s | z_t) \\ q_{t,k'}(z^t) &\geq \max_i MV_{i,t,k'}(z^t) \end{aligned}$$

with strict inequality if all agents are constrained at least once in z^s.

Interest Rates

- Arrow prices are higher than in an otherwise identical economy without solvency constraints in any one of the following cases:
  - constant aggregate endowment,
  - CRRA,
  - quadratic utility.
- Under this conditions, the price of one-period bond is strictly higher in an economy with solvency constraints if at least one agent is constrained in each period.

Premium for aggregate risk

- Aggregate growth, CRRA, constant β.
- Aggregate shock is i.i.d. and independent of the idiosyncratic shock.
- Multiplicative premium on a one-period risky strip:

$$\frac{\sum_{z_{t+1}} d_{t+1}(y_{t+1})\pi(z_{t+1}|z_t)}{\sum_{z_{t+1}} q_t(z^{t+1})d_{t+1}(y_{t+1})} \Big/ \frac{1}{\sum_{z_{t+1}} q_t(z^{t+1})}$$

Prop. The multiplicative premium on a one-period risky strip is the same in an economy with and without participation constraints.

Aggregate growth

- Economy with stochastic growth, constant β, CRRA $u = c^{1-\gamma}/(1-\gamma)$:

$$e_{t+1}(z^{t+1}) = e_t(z^t)\lambda(z_{t+1}) \qquad e_{i,t} = e_t\,\varepsilon_{i,t}$$

- Economy with constant e and state contingent β:

$$\begin{aligned}
\hat{c}_{i,t} &= c_{i,t}/e_t \\
\hat{\pi}(z'|z) &= \frac{\pi(z'|z)\lambda(z')^{1-\gamma}}{\sum_{z'} \pi(z'|z)\lambda(z')^{1-\gamma}} \\
\hat{\beta}(z) &= \beta \sum_{z'} \pi(z'|z)\lambda(z')^{1-\gamma}
\end{aligned}$$

- Resource feasibility and participation constraints are satisfied in one economy iff they are satisfied in the other economy. Moreover, preference orderings are identical.