Dynamic Mechanism Design with Hidden Income and Hidden Actions

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Objective

- General method for computing optimal contracts when the state of the economy is endogeneous.
- Examples:
  - Hidden income and hidden storage.
  - Hidden effort, hidden borrowing and lending, public outcome.

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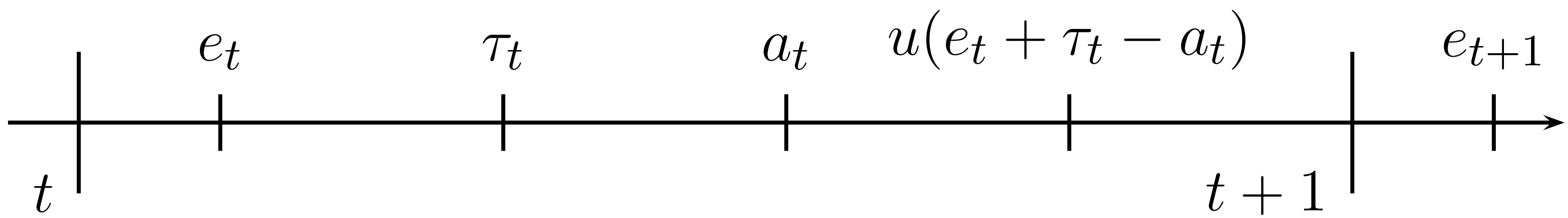
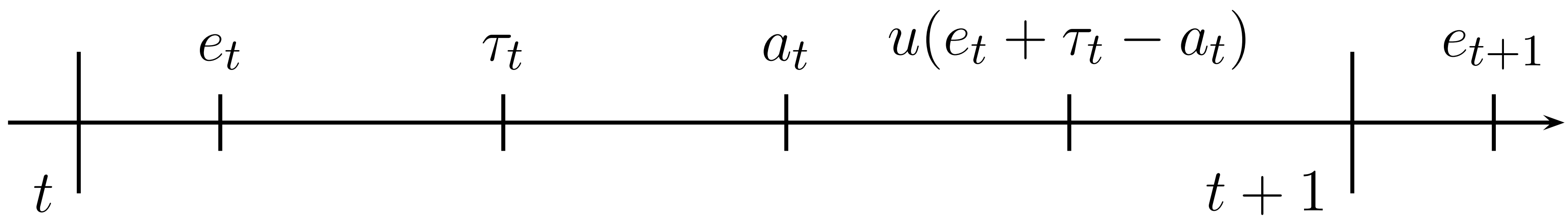
- General method for computing optimal contracts when the state of the economy is endogeneous.
- Examples:
  - Hidden income and hidden storage.
  - Hidden effort, hidden borrowing and lending, public outcome.
- Strategy:
  - General set-up with full history dependence.
  - Recursive formulation with state contingent utility promises.
  - Curse of dimensionality: off-path utility bounds.

General set-up



- $\mu(e|a) > 0$, for all $e \in E$, all $a \in A$ (finite sets)
- History (at time t): $h_t \equiv \{e_t, m_{1t}, \tau_t, m_{2t}, a_t\}$
- Planner knows: $s_t \equiv \{m_{1t}, \tau_t, m_{2t}\}$

General set-up



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- Planner knows: $s_t \equiv \{m_{1t}, \tau_t, m_{2t}\}$
- Planner's outcome function: $\pi(\tau_t, m_{2t}|m_{1t}, s^{t-1})$
- Agent's strategy:
  $\sigma(m_{1t}|e_t, h^{t-1})$ and $\sigma(a_t|e_t, m_{1t}, \tau_t, m_{2t}, h^{t-1})$
- Agent's expected utility:
  $U(\pi, \sigma) = \sum_{t=0}^{\infty} \beta^t [\sum_{H_t} p(h^t|\pi, \sigma) u(e_t + \tau_t - a_t)]$

Optimal equilibrium

- An equilibrium $\{\pi, \sigma\}$ satisfies:

$$U(\pi, \hat{\sigma}|h^k) \leq U(\pi, \sigma|h^k), \forall \hat{\sigma}, h^k \quad (1)$$

$$U(\pi, \sigma) \geq W_0 \quad (2)$$

- A feasible allocation is a probability distribution over income, transfers and actions generated by an equilibrium.
- An optimal equilibrium solves the planner's problem:

$$\max V(\pi, \sigma) \equiv \sum_{t=0}^{\infty} Q^t [\sum_{H_t} p(h^t|\pi, \sigma)(-\tau_t)]$$

$$s.t \;\; (1) \;\; and \;\; (2)$$

Deriving a recursive formulation (1)

- Let $M_1 = E$, $M_2 = A$ and $s_t \equiv \{e_t, \tau_t, a_t\}$.
- Let $\delta_e(h^{t-1}, e_t)$ and $\delta_a(h^{t-1}, e_t, \tau_t, a_t)$ be fully history-dependent deviation strategy δ.
- An equilibrium outcome function under truth telling and obedience satisfies $\forall \delta, s^k, e_{k+1}$:

$$\sum_{t=0}^{\infty} \beta^t [\sum_{S^t} p(s^t|\pi) u(e_t + \tau_t - a_t)] \geq W_0 \tag{3}$$

$$\sum_{t=k+1}^{\infty} \beta^t [\sum_{H^t} p(h^t|\pi, \delta, s^k, e_{k+1}) u(e_t + \tau_t - \delta_a)] \leq \sum_{t=k+1}^{\infty} \beta^t [\sum_{S^t} p(s^t|\pi, s^k, e_{k+1}) u(e_t + \tau_t - a_t))] \tag{4}$$

Deriving a recursive formulation (2)

- (Revelation Principle) Any allocation that is feasible in the general mechanism is also feasible in the truth-telling and obedience mechanism.
- (Auxiliary problem) Promise keeping constraint for each possible initial income e_0:

$$\sum_{T,A} \pi(\tau_0, a_0 | \boldsymbol{w_0}, e_0)[u(e_0 + \tau_0 - a_0) + \sum_{t=1}^{\infty} \beta^t [p(s^t | \pi, s_0) u(e_t + \tau_t - a_t)]] = w_0(e_0) \tag{5}$$

Deriving a recursive formulation (3)

- Let $\boldsymbol{W}$ be the set of all vectors $\boldsymbol{w}_0 \in R^{\#E}$ that satisfy (4) and (5) for some outcome function π.
- (Principle of optimality)
  For all $\boldsymbol{w}_0 \in \boldsymbol{W}$ and $e_0 \in E$, and for any s^{k-1} and e_k, there is an optimal contract π^* such that the remaining contract from s^{k-1} and e_k onward is an optimal contract for the auxiliary problem with $e_0 = e_k$ and $\boldsymbol{w}_0 = \boldsymbol{w}(s^{k-1}, \pi^*)$

Program 1. W finite set

$$V(\boldsymbol{w}, e) = max_{\pi} \sum_{T,A,\boldsymbol{W'}} \pi(\tau, a, \boldsymbol{w'}|\boldsymbol{w}, e)[-\tau + Q \sum_{E} \mu(e'|a) V(\boldsymbol{w'}, e')] \ s.t.$$

$$\sum_{T,A,\boldsymbol{W'}} \pi(\tau, a, \boldsymbol{w'}|\boldsymbol{w}, e)[u(e, \tau, a) + \beta \sum_{E} \mu(e'|a) w'(e')] = w(e) \tag{6}$$

$$\sum_{\boldsymbol{W'}} \pi(\boldsymbol{w'})[u(e, \tau, \hat{a}) + \beta \sum_{E} \mu(e'|\hat{a}) w'(e')] \leq \sum_{\boldsymbol{W'}} \pi(\boldsymbol{w'})[u(e, \tau, a) + \beta \sum_{E} \mu(e'|a) w'(e')] \tag{7}$$

Curse of dimensionality

$$\sum_{T,A,\boldsymbol{W}'} \pi(\tau, a, \boldsymbol{w}'|\boldsymbol{w}, e)[u(\hat{e}, \tau, \delta(\tau, a)) + \beta \sum_{E} \mu(e'|\delta(\tau, a))w'(e')] \leq w(\hat{e}) \tag{8}$$

- Number of variables: $\#T \times \#A \times \#\boldsymbol{W}$.
- Number of obedience constraints (7): $\#T \times \#A \times (\#A - 1)$.
- Number of truth telling constraints (8): $(\#E - 1)(\#A)^{(\#T \times \#A)}$.

Imposing off-path utility bounds

$$V(\boldsymbol{w}, e) = max_{\pi \geq 0, v} \sum_{T,A,\boldsymbol{W'}} \pi(\tau, a, \boldsymbol{w'}|\boldsymbol{w}, e)[-\tau + Q \sum_{E} \mu(e'|a) V(\boldsymbol{w'}, e')]$$

$$s.t. \quad (6), \quad (7) \quad and$$

$$\sum_{\boldsymbol{W'}} \pi(\boldsymbol{w'})[u(\hat{e}, \hat{a}) + \beta \sum_{E} \mu(e'|\hat{a}) w'(e')] \leq v(\hat{e}, e, \tau, a) \tag{9}$$

$$\forall \hat{e}, \tau, a, \hat{a}$$

$$\sum_{T,A} v(\hat{e}, e, \tau, a) \leq w(\hat{e}) \ , \quad \forall \hat{e} \tag{10}$$

Avoiding curse of dimensionality

- Number of variables:
  $\#T \times \#A \times \#\boldsymbol{W} + (\#E - 1) \times T \times (\#A)$.
- Number of obedience constraints (7):
  $\#T \times \#A \times (\#A - 1)$.
- Number of constraint to implement bounds:
  $(\#E - 1) \times T \times (\#A)^2$.
- Number of truth telling constraints: $(\#E - 1)$.
- Not sufficient to check for local deviations.

Hidden income and hidden storage (1)

$$\beta = Q = 1.05^{-1} \;\; \#E = 2 \;\; \#A = 3 \;\; \#T_1 = T_2 = 50 \;\; T_3 = 4000$$

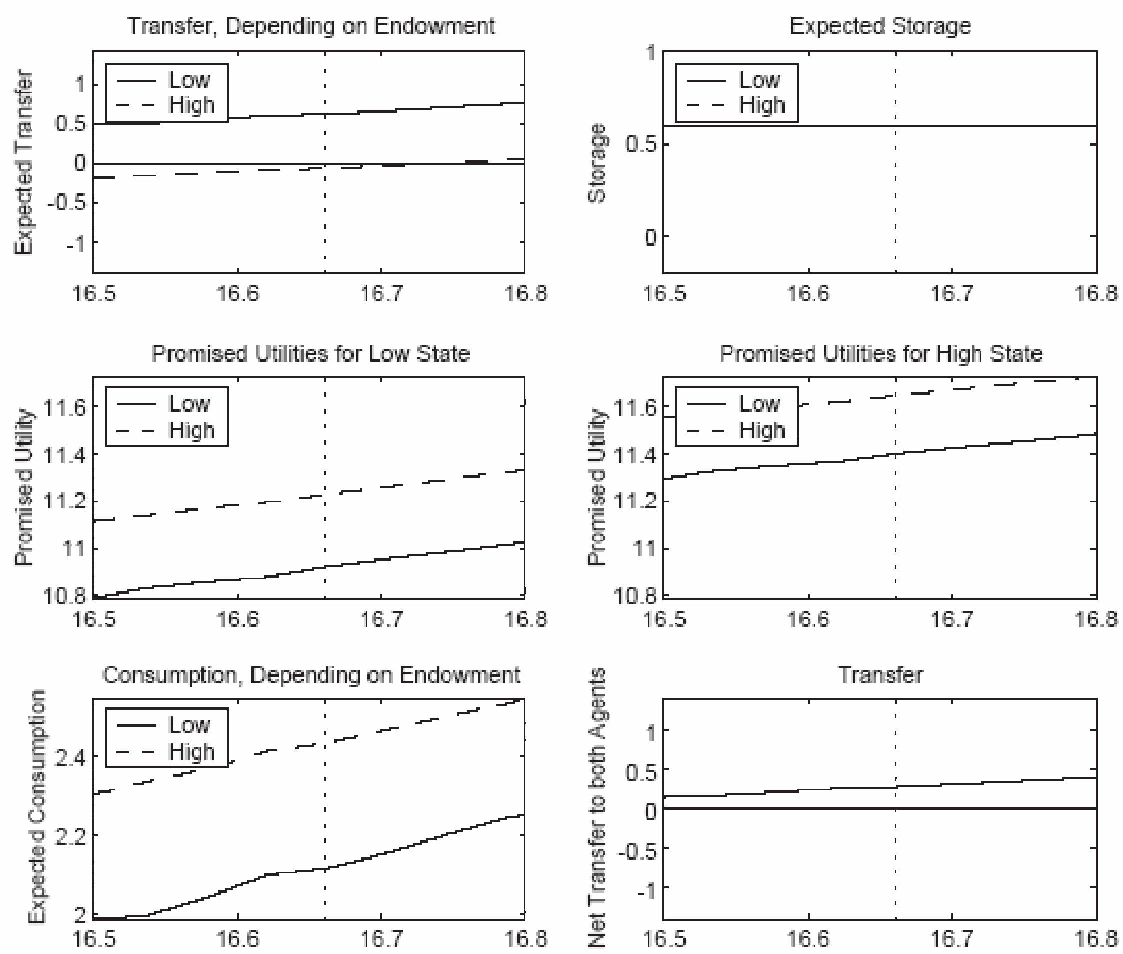

Fig. 3. Policy functions, $R = 1.1$.

Hidden income and hidden storage (2)

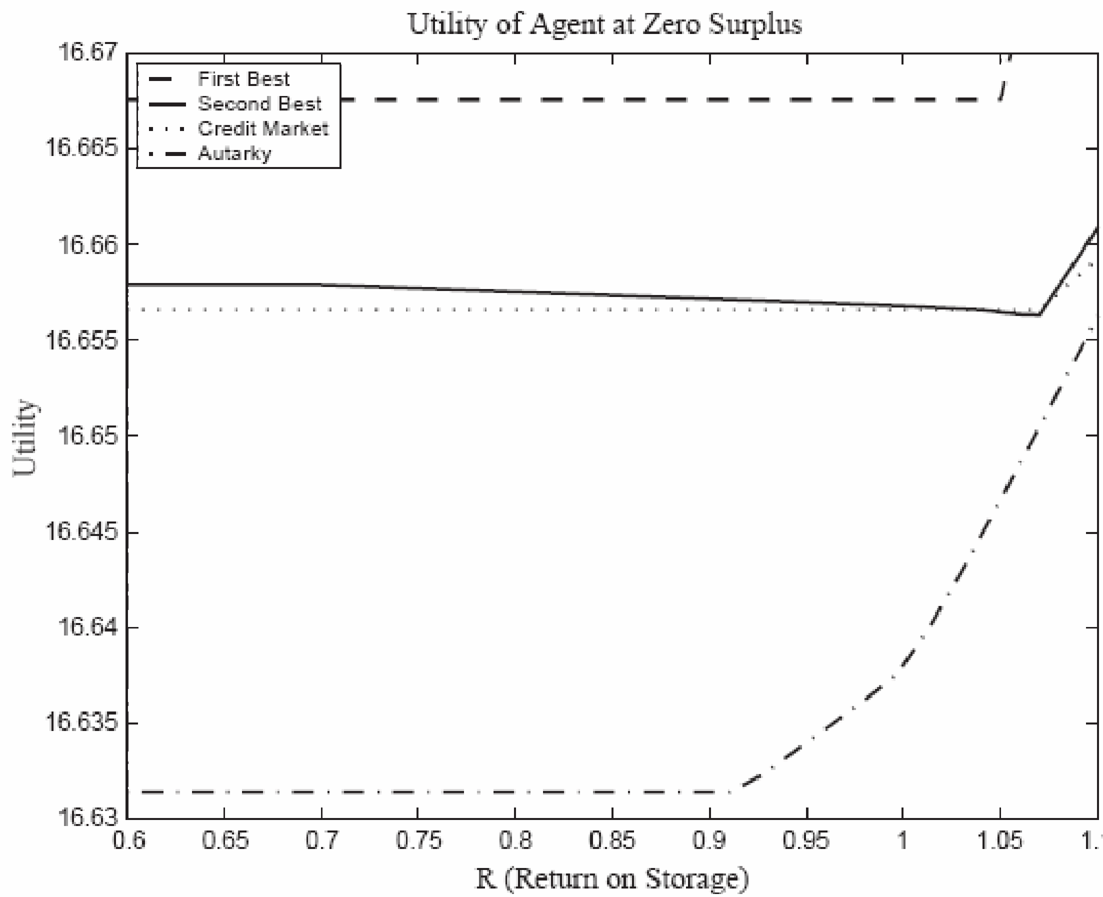

Fig. 4. Utility of the agent with full information, private information, credit-market access, and autarky.