Robust Equilibrium Yield Curves

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Introduction

- Historically, the yield on bonds has been higher, the longer the maturity of the bond is.
- If consumption growth rates are positively auto-correlated, we would expect to see just the oposite. (Bonds then act as a hedge against consumption risk)
- To what extent can the term premium on bonds be explained by investors' wanting models which are robust to misspecification?

Endowment process

Reference model equals true model, but agent do not trust it completely.

$$dC = \mu C \; dt + \sqrt{v} C \; dB$$
$$dv = -k(v - \bar{v}) + \sqrt{v} \; dB$$

Same univariate Brownian motion driving both consumption growth and variance!

Worst case model

His worst case distribution $\mathbb{Q}$ for the endowment process comes from

$$\inf_{\mathbb{Q}} \left\{ E_t^{\mathbb{Q}} \left[\int t^{\infty} e^{-\rho(s-t)} \log C_s ds \right] + \theta \mathcal{R}_t(\mathbb{Q}) \right\},$$

$\mathcal{R}_t(\mathbb{Q})$ is an entropy penalty. (Higher the further $\mathbb{Q}$ is from $\mathbb{P}$.)

Control of the evil agent

The control of the evil agent is h, a time-varying distortion to the mean drift rate of the Brownian motion. The distorted distribution $\mathbb{Q}$ is linked to $\mathbb{P}$ through the Radon-Nikodym derivative:

$$\xi_t^{\mathbb{Q}} = \exp\left(\int_0^t h_s \, dB_s - \frac{1}{2}\int_0^t h_s^2 \, ds\right)$$

The distortion to the drift of C is $\sqrt{v}h$.

Distortion to drift is linear in $v(t)$

Intuition (1 period model, log utility, log normal endowment):

$$\log C \sim N(\mu, \sigma^2)$$
$$E^{\mathbb{Q}}[\log C] = \mu + h$$
$$\mathcal{R}(\mathbb{Q}) = \int \log \frac{d\mathbb{Q}}{d\mathbb{P}} d\mathbb{Q} = \frac{h^2}{2\sigma^2}$$

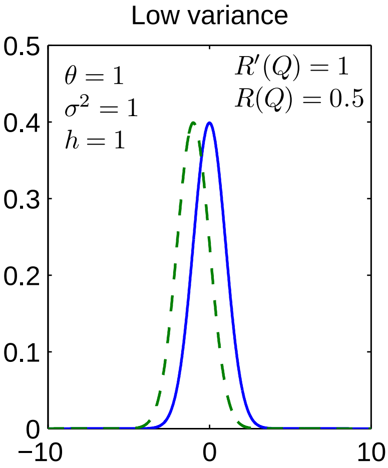
The worst case distribution is given by the shift of the mean which minimizes $E^{\mathbb{Q}}[\log C] + \theta\mathcal{R}(\mathbb{Q})$:

$$\frac{d}{dh}E^{\mathbb{Q}}[\log C] + \theta\frac{d}{dh}\mathcal{R}(\mathbb{Q}) = 1 + \theta\frac{h}{\sigma^2}$$

Minimizing h is thus

$$h = -\frac{\sigma^2}{\theta}$$

Graphical example



Short rate with Robustness

$$r_t = \rho + \underbrace{\mu + \sqrt{v_t} h_t}_{\text{Worst case drift}} - v_t$$

Short rate decrease with positive consumption innovations

$$C(t+\Delta) - C(t) \approx \sqrt{v(t)} \times \Big(B(t+\Delta) - B(t) \Big)$$

$$v(t+\Delta) - C(t) \approx \sigma_v \sqrt{v(t)} \times \Big(B(t+\Delta) - B(t) \Big)$$

If $(B(t+\Delta) - B(t)) > 0$, then the conditional variance increases.
Drives down interest rates by lowering the drift under $\mathbb{Q}$

Mechanism: Countercyclical interest rate

- Drift contamination increases with conditional standard deviation
$$h_t \propto -\sqrt{v_t}$$
- Conditional standard deviation increase with positive consumption shocks (from process assumptions).
- Through the drift contamination this drives down expected consumption growth rates. (Short rate drops.)
- Bonds rally.
- Bonds procyclical, consumer require a premium to hold them.