Incomplete Markets and Volatility

Laurent Calvet

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Setup-Basic Features

- time discrete $t = 1, 2, ..., T \leq \infty$, $h = 1, ..., H$ agents, one good

- underlying space $(\Omega, \mathcal{F}, \mathcal{P})$, endowment process $\left\{\tilde{e}_t^h\right\}_{t=0}^T$, independent across time

- *deterministic* mean endowment $\tilde{e}_t = \frac{\sum_h \tilde{e}_t^h}{H} = e_t \;\; \forall t = 1, ..., T$

- Additive separable utility $E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$

  CARA period utility $u(c) = -\frac{1}{A} \exp(-Ac)$

Financial structure-CARA-Normal Setup

- sequential markets, one riskless asset and $N(t)$ risky real assets, in zero net supply

- prices at t (in terms of the consumption good): $\pi(t) = \left(\pi_0(t), \pi_1(t), ..., \pi_{N(t)}\right.$

- riskless rate $R_t \equiv \frac{1}{\pi_0(t)}$

- $S(t)$: subspace of $L^2(\Omega)$ spanned by the $N(t)+1$ assets

- payoff vector $\tilde{A}_{t+1} = \left(a_{0,t+1}, \tilde{a}_{1,t+1}, ..., \tilde{a}_{N(t),t+1}\right)'$ jointly normal with individual endowments

- pick an orthonormal basis s.t. $a_{0,t+1} \equiv 1$, $E\tilde{a}_{i,t+1} = 0$ for $i = 1, ..., N(t)$

- Thus $E\tilde{a}_{i,t+1}^2 = 1$ and $E\tilde{a}_{i,t+1}\tilde{a}_{j,t+1} = 0$

Information structure/Index of incompleteness

- *symmetric information*, Filtration $F_k = \left\{ \left\{ \tilde{e}_t^h \right\}_{h=1}^H , \tilde{A}_t \right\}_{t=0}^k , k = 0, 1, ..., T$

- project individual income on asset span $S(t)$

$$\tilde{e}_{t+1}^h = E_t \tilde{e}_{t+1}^h + \sum_{i=1}^{N(t)} \gamma_{it}^h \tilde{a}_{i,t+1} + \tilde{\varepsilon}_{t+1}^h, \text{ where } \gamma_{it}^h \equiv Cov_t \left(\tilde{a}_{i,t+1}, \tilde{e}_{t+1}^h \right)$$

- $\tilde{\varepsilon}_{t+1}^h$:*undiversifiable* risk, Index of Market Incompleteness

$$V(t) \equiv \frac{A^2}{H} \sum_{h=1}^{H} Var_t \left(\tilde{\varepsilon}_{t+1}^h \right)$$

Equilibrium

- Let $\theta_t^h = \left(\theta_{0,t}^h, \theta_{1,t}^h, ..., \theta_{N(t),t}^h\right)$ be the portfolio of agent h.

- *Admissible* plan $\left\{c_t^h, \theta_t^h, W_t^h\right\}_{t=0}^T$ that satisfies the *budget* constraints

$$c_t^h + \pi(t) \cdot \theta_t^h = W_t^h \text{ and } W_{t+1}^h = e_{t+1}^h + A_{t+1} \cdot \theta_t^h$$

- Equilibrium $\{\pi(t)\}_{t=0}^T$ and a collection of admissible plans $\left\{\left\{c_t^h, \theta_t^h, W_t^h\right\}_{t=0}^T\right\}_h$

s.t a) each admissible plan is optimal b) markets clear $\frac{\sum_{h=1}^H c_t^h}{H} = e_t \ \forall t$

and $\frac{\sum_{h=1}^H \theta_t^h}{H} = 0 \ \forall t$

Decision Problem

- state for the DM W_t^h (because of iid endowment)

- Value function $J^h\left(W_t^h, t\right)$ exponential because of CARA , linear desicion rules

- *Theorem 1*: Optimal plan $\left(c_t^h, \theta_t^h\right)$ satisfies

  $c_t^h = a_t W_t^h + b_t^h$

  $\theta_{it}^h = -\gamma_{it}^h - R_t \pi_i(t) / A a_{t+1}, i = 1, ..., N(t)$

  $\theta_{0t}^h = \frac{a_t}{a_{t+1}} W_t^h - R_t b_t^h - R_t \sum_{i=1}^{N(t)} \pi_i(t) \theta_{it}^h$

where $\{a\}_{t=0}^{T}$ satisfies the recursion

- $$\frac{1}{a_t} = 1 + \frac{1}{R_t a_{t+1}}$$

with terminal condition $a_T = 1$ and $b_T^h = 0$ for $T < \infty$ and no terminal condition for $T = \infty$

- Iterating on the MPC

$$\begin{aligned}\frac{1}{a_t} &= 1+\frac{1}{R_t}+\frac{1}{R_t R_{t+1}}+...+\frac{1}{R_t R_{t+1}...R_{T-1}} \\ &= 1+\pi_L(t)\end{aligned}$$

where $\pi_L(t)$: price of a perpetuity

- Partial equilibrium: Positive feedback between future and current MPC

General Equilibrium $T < \infty$

- Unique equilibrium with no risk premium $\pi_i(t) = 0, i = 1, ..., N(t)$

$$\ln R_t = \ln\left(\frac{1}{\beta}\right) - \frac{1}{2}a_{t+1}^2 V(t) + A(e_{t+1} - e_t)$$

- Note that: if markets complete ($V = 0$) and stationary economy ($e_{t+1} = e_t$), then $R_t = \frac{1}{\beta}$

- GE effect: negative relationship between future and current interest rates because of the precautionary motive

- Increase in MPC_{t+1} may lead to a decrease in the MPC_t

- Stationary economy ($e_{t+1} = e_t$) and $V(t) = V$. Represent the equilibrium as IFS

$$a_t = F(a_{t+1}) \text{ and } a_T = 1$$

where $F(a) = \frac{1}{1+\beta a^{-1}\exp\left(a^2 V/2\right)}$

- *Theorem:*Let $\mathcal{E}_{V,\beta} \equiv \{E_T\}_{T=0}^{\infty}$, and let $\left\{a_0^T = F^T(1)\right\}_{T=0}^{\infty}$ be the sequence of time 0 MPC. Then $\exists$ open set U s.t. $\forall (V,\beta) \in U$, $\left\{a_0^T\right\}_{T=0}^{\infty}$ *does not* converge to a s.s.(for intermediate incompleteness and sufficient impatience)

- *Theorem*: For a stationary economy with $T = \infty$, we can either have a unique equilibrium/s.s that satisfies $F(\bar{a}) = \bar{a}$, or a continuum of equilibria, that include also a cycle of order 2.

Let for example $\beta = 0.2$ and $T = 100$