Six anomalies looking for a model:
A consumption based explanation of international finance puzzles

Riccardo Colacito
Six Puzzles

- Backus-Smith anomaly
  
  \[ \text{model} : \quad \Delta e_{t+1} = m_{t+1}^* - m_{t+1} \]
  
  \[ \text{data} : \quad \text{corr} (\Delta c - \Delta c^*, \Delta e) = 0.072 \]

- Brandt, Cochrane and Stanta-Clara puzzle
  
  \[ \text{model} : \quad \text{var} (\Delta e_{t+1}) = \text{var} (m_{t+1}^*) + \text{var} (m_{t+1}) - 2 \text{cov} (m_{t+1}^*, m_{t+1}) \]
  
  \[ \text{data} : \quad \text{std} (\Delta e) = 0.11, \text{std} (m^\text{UK}) = 0.39, \text{std} (m^\text{US}) = 0.37 \]

- UIP puzzle
  
  \[ \text{model} : \quad E_t [\Delta e_{t+1}] = r_t - r_t^* - \frac{1}{2} \text{var}_t (m_{t+1}^*) + \frac{1}{2} \text{var}_t (m_{t+1}) \]
  
  \[ \text{data} : \quad \Delta e_{t+1} = \alpha + \beta (r_t - r_t^*) + \zeta_{t+1} : \beta = 0 \]
Six Puzzles

- equity premium puzzle (average Sharpe ratios 0.38 for G7)
- low correlation in fundamentals, but high correlation in stock returns, and
- higher but not perfect correlation in risk-free rates

\[
\begin{align*}
\text{corr}(\Delta c, \Delta c^*) & = 0.125 \\
\text{corr}(\Delta d, \Delta d^*) & = 0.108 \\
\text{corr}(r_{stk}, r_{stk}^*) & = 0.489 \\
\text{corr}(r_{rf}, r_{rf}^*) & = 0.678
\end{align*}
\]
Model

- Complete markets
- Representative agents have risk-sensitive preferences:

\[
U_t = (1 - \delta) \log C_t + \delta \theta \log E_t \exp \left( \frac{U_{t+1}}{\theta} \right)
\]

\[
U^*_t = (1 - \delta) \log C^*_t + \delta \theta \log E_t \exp \left( \frac{U^*_{t+1}}{\theta} \right)
\]

with \( \theta = \frac{1}{1-\gamma} \).
Model

Long-run risk in consumption:

\[
\log C_{t+1} - \log C_t = \Delta c_{t+1} = \mu_c + \lambda_1 z_{1,t} + \lambda_2 z_{2,t} + \lambda_t^{1/2} \varepsilon_{t+1} \\
\log C^*_t - \log C_t = \Delta c^*_{t+1} = \mu_c + \lambda^*_1 z_{1,t} + \lambda^*_2 z_{2,t} + \lambda^*_t^{1/2} \varepsilon^*_{t+1}
\]

\[
z_{1,t} = \rho_1 z_{1,t-1} + \lambda_t^{1/2} \varepsilon_{1,t} \\
z_{2,t} = \rho_2 z_{2,t-1} + \lambda^*_t^{1/2} \varepsilon_{2,t} \\
\lambda_t = \sigma(1 - \rho_\lambda) + \rho_\lambda \lambda_{t-1} + \phi_\lambda \lambda_{t-1} \varepsilon_\lambda,t \\
\lambda_t^* = \sigma(1 - \rho_\lambda) + \rho_\lambda \lambda^*_t + \phi_\lambda \lambda^*_{t-1} \varepsilon^*_\lambda,t
\]

Dividend dynamics:

\[
\Delta d_{t+1} = \mu_d + \lambda_{d1} z_{1,t} + \lambda_{d2} z_{2,t} + \lambda_t^{1/2} \varepsilon_{d,t+1} \\
\Delta d^*_{t+1} = \mu_d + \lambda^*_{d1} z_{1,t} + \lambda^*_{d2} z_{2,t} + \lambda^*_t^{1/2} \varepsilon^*_{d,t+1}
\]
## Baseline Calibration

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk aversion</td>
<td>$\gamma$ 8.000</td>
</tr>
<tr>
<td>Subjective discount factor</td>
<td>$\delta$ 0.991</td>
</tr>
<tr>
<td>Average consumption growth</td>
<td>$\mu_c$ 0.001</td>
</tr>
<tr>
<td>Average dividend growth</td>
<td>$\mu_d$ 0.001</td>
</tr>
<tr>
<td>Loading on $z_{1,t}$ in $\Delta c_{t+1}$</td>
<td>$\lambda_1$ 5.500</td>
</tr>
<tr>
<td>Loading on $z_{2,t}$ in $\Delta c_{t+1}$</td>
<td>$\lambda_2$ 1.000</td>
</tr>
<tr>
<td>Loading on $z_{1,t}$ in $\Delta c^*_t$</td>
<td>$\lambda_1^*$ 1.000</td>
</tr>
<tr>
<td>Loading on $z_{2,t}$ in $\Delta c^*_t$</td>
<td>$\lambda_2^*$ 5.500</td>
</tr>
<tr>
<td>Loading on $z_{1,t}$ in $\Delta c_{t+1}$</td>
<td>$\lambda_{d1}$ 24.000</td>
</tr>
<tr>
<td>Loading on $z_{2,t}$ in $\Delta c_{t+1}$</td>
<td>$\lambda_{d2}$ 2.000</td>
</tr>
<tr>
<td>Loading on $z_{1,t}$ in $\Delta c^*_t$</td>
<td>$\lambda_{d1}^*$ 2.000</td>
</tr>
<tr>
<td>Loading on $z_{2,t}$ in $\Delta c^*_t$</td>
<td>$\lambda_{d2}^*$ 24.000</td>
</tr>
<tr>
<td>Persistence of first predictive factor</td>
<td>$\rho_1$ 0.987</td>
</tr>
<tr>
<td>Persistence of second predictive factor</td>
<td>$\rho_2$ 0.987</td>
</tr>
<tr>
<td>Persistence of stochastic volatility</td>
<td>$\rho_\lambda$ 0.950</td>
</tr>
<tr>
<td>Predictive factors to consumption growth</td>
<td>$\varphi_c$ 8.80E-03</td>
</tr>
<tr>
<td>standard error ratio</td>
<td></td>
</tr>
<tr>
<td>Dividend to consumption growth standard error</td>
<td>$\varphi_d$ 4.500</td>
</tr>
<tr>
<td>Stochastic volatility to consumption growth</td>
<td>$\varphi_\lambda$ 3.50E-04</td>
</tr>
<tr>
<td>Standard error of the shock to consumption growth</td>
<td>$\sigma$ 0.006</td>
</tr>
</tbody>
</table>

### Correlations

<table>
<thead>
<tr>
<th>Correlation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Predictive factors</td>
<td>$\rho_{1,2}$ 0.600</td>
</tr>
<tr>
<td>Predictive factors and consumption growth</td>
<td>$\rho_{1,c} = \rho_{2,c^*}$ 0.060</td>
</tr>
<tr>
<td>Predictive factors and consumption growth</td>
<td>$\rho_{1,c^*} = \rho_{2,c}$ 0.600</td>
</tr>
<tr>
<td>Predictive factors and stochastic volatility</td>
<td>$\rho_{1,\lambda} = \rho_{2,\lambda^<em>} = -\rho_{2,\lambda} = -\rho_{1,\lambda^</em>}$ -0.285</td>
</tr>
<tr>
<td>Consumption growth and stochastic volatility</td>
<td>$\rho_{c,\lambda} = \rho_{c^<em>,\lambda^</em>} = -\rho_{c,\lambda^<em>} = -\rho_{c^</em>,\lambda}$ 0.470</td>
</tr>
<tr>
<td>Dividend growths</td>
<td>$\rho_{d,d^*}$ 0.050</td>
</tr>
<tr>
<td>Consumption growths</td>
<td>$\rho_{c,c^*}$ 0.100</td>
</tr>
</tbody>
</table>
Baseline calibration - long-run risk

\[ \log C_{t+1} - \log C_t = \Delta c_{t+1} = \mu_c + 5.5z_{1,t} + z_{2,t} + \lambda_t^{1/2} \varepsilon_{t+1} \]

\[ \log C^*_{t+1} - \log C^*_t = \Delta c^*_{t+1} = \mu_c + z_{1,t} + 5.5z_{2,t} + \lambda_t^{*1/2} \varepsilon^*_{t+1} \]

\[ z_{1,t} = 0.987z_{1,t-1} + \lambda_{t-1}^{1/2} \varepsilon_{1,t} \]

\[ z_{2,t} = 0.987z_{2,t-1} + \lambda_{t-1}^{*1/2} \varepsilon_{2,t} \]

\[ \lambda_t = 0.006(1 - 0.95) + 0.95\lambda_{t-1} + 0.00035\lambda_{t-1} \varepsilon_{\lambda,t} \]

\[ \lambda^*_t = 0.006(1 - 0.95) + 0.95\lambda^*_{t-1} + 0.00035\lambda^*_{t-1} \varepsilon^*_{\lambda,t} \]
## Benchmark Result

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility of exchange rate growth</td>
<td>10.696</td>
<td>10.929</td>
<td>(9.105, 12.954)</td>
</tr>
<tr>
<td>UIP regression slope</td>
<td>-0.136</td>
<td>-0.062</td>
<td>(-0.609, 0.458)</td>
</tr>
<tr>
<td>Sharpe ratio</td>
<td>37.781</td>
<td>40.564</td>
<td>(2.783, 76.447)</td>
</tr>
<tr>
<td>Backus and Smith anomaly</td>
<td>0.072</td>
<td>-0.056</td>
<td>(-0.279, 0.165)</td>
</tr>
<tr>
<td>Correlation of risk-free rates</td>
<td>0.678</td>
<td>0.793</td>
<td>(0.189, 0.954)</td>
</tr>
<tr>
<td>Correlation of excess returns</td>
<td>0.489</td>
<td>0.422</td>
<td>(0.232, 0.578)</td>
</tr>
<tr>
<td>Volatility of consumption growth</td>
<td>1.937</td>
<td>1.909</td>
<td>(1.547, 2.388)</td>
</tr>
<tr>
<td>AC(1)</td>
<td>0.203</td>
<td>0.347</td>
<td>(0.081, 0.598)</td>
</tr>
<tr>
<td>AC(2)</td>
<td>0.191</td>
<td>0.158</td>
<td>(-0.154, 0.484)</td>
</tr>
<tr>
<td>AC(3)</td>
<td>0.158</td>
<td>0.140</td>
<td>(-0.161, 0.473)</td>
</tr>
<tr>
<td>AC(4)</td>
<td>0.014</td>
<td>0.125</td>
<td>(-0.141, 0.448)</td>
</tr>
<tr>
<td>Correlation of consumption growths</td>
<td>0.125</td>
<td>0.247</td>
<td>(-0.032, 0.546)</td>
</tr>
<tr>
<td>AC(1)</td>
<td>0.158</td>
<td>0.109</td>
<td>(-0.142, 0.384)</td>
</tr>
<tr>
<td>AC(2)</td>
<td>0.093</td>
<td>0.088</td>
<td>(-0.161, 0.376)</td>
</tr>
<tr>
<td>AC(3)</td>
<td>0.086</td>
<td>0.087</td>
<td>(-0.161, 0.358)</td>
</tr>
<tr>
<td>AC(4)</td>
<td>0.182</td>
<td>0.075</td>
<td>(-0.164, 0.356)</td>
</tr>
<tr>
<td>Correlation of dividend growths</td>
<td>0.108</td>
<td>0.135</td>
<td>(-0.113, 0.380)</td>
</tr>
</tbody>
</table>
Solution of the model

Log discount rate can be solved as follows:

\[
\begin{align*}
 m_{t+1} &= -v_{0,m} - v_m s'_t + \frac{1}{\theta} \left( \tilde{v}_1 \lambda_t^{1/2} + v_2 \lambda_t^{*1/2} \right) \eta'_t \\
 m_{t+1}^* &= -v_{0,m} - v_m^* s'_t + \frac{1}{\theta} \left( v_1^* \lambda_t^{1/2} + \tilde{v}_2 \lambda_t^{*1/2} \right) \eta'_t \\
 s_t &= [z_{1,t}, z_{2,t}, \lambda_t, \lambda_t^*, h_t] \\
 h_t &= \lambda_t^{1/2} \lambda_t^{*1/2} \\
 \eta_t &= [\varepsilon_{1,t}, \varepsilon_{2,t}, \varepsilon_c,t, \varepsilon_c^*,t, \varepsilon_\lambda,t, \varepsilon_\lambda^*,t]
\end{align*}
\]

Solution method
Volatility of real exchange rate changes

- Complete market →

\[ \Delta e_{t+1} = m^*_{t+1} - m_{t+1} \]
\[ = (v_m - v^*_m) s'_t + \frac{1}{\theta} \left( (v^*_1 - \tilde{v}_1) \lambda_t^{1/2} + (\tilde{v}^*_2 - v_2) \lambda^*_t^{1/2} \right) \eta'_{t+1} \]

- Key: pick correlation matrix of the shocks carefully.
Correlation of risk free rates

- Let's shut down uncertainty shocks for now.
- Short rates are:

\[
\begin{align*}
    r_t &= \bar{r}_t + \lambda_1 z_{1,t} + \lambda_2 z_{2,t} \\
    r^*_t &= \bar{r}^*_t + \lambda^*_1 z_{1,t} + \lambda^*_2 z_{2,t}
\end{align*}
\]

- Key: two serially correlated shocks with proper correlation.
Use Campbell-Shiller decomposition:

\[
\begin{align*}
    r_{d,t+1} &= k_0 + \Delta d_{t+1} - k_1 dp_{t+1} + dp_t \\
    dp_t &= \bar{dp} + \frac{\lambda_1 - \lambda_{d1}}{1 - k_1 \rho_1} z_{1,t} + \frac{\lambda_2 - \lambda_{d2}}{1 - k_1 \rho_2} z_{2,t} \\
    dp^*_t &= \bar{dp}^* + \frac{\lambda_2 - \lambda_{d2}}{1 - k_1 \rho_2} z_{1,t} + \frac{\lambda_1 - \lambda_{d1}}{1 - k_1 \rho_1} z_{2,t}
\end{align*}
\]

- Key: pick \( \text{corr}(z_1, z_2) \) carefully.
Backus-Smith puzzle

- Keep uncertainty shocks shut down.
- Innovations are as follows

\[
\begin{align*}
\Delta e &= \beta_1 \varepsilon_1 - \beta_2 \varepsilon_2 + \lambda \varepsilon - \lambda^* \varepsilon^* \\
\Delta c - \Delta c^* &= \lambda \varepsilon - \lambda^* \varepsilon^*
\end{align*}
\]

- asymmetric loadings \( \rightarrow \beta_1 \neq \beta_2 \)
- Key: correlation: \( \rho_{1,c} = \rho_{2,c^*} = 0.06, \rho_{1,c^*} = \rho_{2,c} = 0.6 \)
Let's turn on uncertainty shocks again.

Solve for slope explicitly:

**UIP regression**
\[ \Delta e_{t+1} = \alpha + \beta (r_t - r_t^*) + \xi_{t+1} \]

**Model counterpart**
\[ m_{t+1}^* - m_{t+1} = \alpha + \beta \left( p_t + \frac{1}{2} q_t \right) + \zeta_{t+1} \]

\[ \hat{\beta} = \frac{\text{Var}(p_t) + \frac{1}{2} \text{Cov}(p_t, q_t)}{\text{Var}(p_t + \frac{1}{2} q_t)} \]

where

\[ p_t = E_t m_{t+1}^* - E_t m_{t+1} = A_1 z_{1,t} - A_2 z_{2,t} + B_1 \lambda_t - B_2 \lambda_t^* \]
\[ q_t = \text{Var}_t m_{t+1}^* - \text{Var}_t m_{t+1} = -B_1 \lambda_t + B_2 \lambda_t^* \]

Key: get $\text{Cov}(p_t, q_t)$ sufficiently negative.
Stochastic Volatility

\[ \sigma(e_{t+1}/e_t) \]

\[ \text{Sharpe Ratio} \]

\[ \text{Correl. risk-free rates} \]

\[ \text{Correl. excess returns} \]

\[ \text{corr}(\Delta c, \Delta c^* e_{t+1}/e_t) \]

\[ \beta_{Up} \]
Long-run risk
Conclusion

- It reverse-engineers a plausible consumption process, builds a complete market two-country model with risk-sensitive preferences and long run risk. It can explain the six puzzles.