“Time-Varying Risk, Interest Rates, and Exchange Rates in General Equilibrium”
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Introduction

• Consider the risk, in nominal terms faced by an investor in the US choosing between bonds denominated in either dollars or euros.
• If $e_t$ is the exchange rate between euros and dollars then the risk premium, in logs, is equal to the expected log dollar return in euro bond minus the log dollar return on a dollar bond

$$p_t = i^* + E_t \log e_{t+1} - \log e_t - i_t$$

where $i^*$ and $i_t$ are the logs of the euro and dollar gross interest rates.
• Nominal exchange rates are viewed to be roughly random walks, so the expected depreciation of the currency, $E_t \log e_{t+1} - \log e_t$ is roughly constant.
• The observed variation in $i^*_t - i_t$ is thus almost entirely do to variation in the risk premia.
A more nuanced view of the data finds that when a currency’s interest rate is high, that currency is expected to appreciate.

This is the *forward premium anomaly* and goes against intuition as one would expect investors would demand a higher interest rate on currencies which are expected to fall in value.

Documented by the regression

\[ \log e_{t+1} - \log e_t = a + b(i_t - i_t^*) + u_{t+1} \]

such regressions usually yield estimates of $b$ that are zero or negative.

The contribution of this paper is to generate a simple general equilibrium monetary model which creates both a variance in risk premium and the forward premium anomaly.

This is done by allowing the asset market to be segmented, only a fraction of the model’s agents choose to participate.
Setup

- Two country, cash-in-advance economy with an infinite number of periods $t = 0, 1, 2, \ldots$. 

• Each country has a government and a continuum of households of measure one.
• Households in the home country use home currency to purchase a home good, and similar for foreign country.
• Trade occurs in three locations: an asset market available to both countries and one goods market in each country.
• In the asset market agents trade a complete set of state contingent bonds and governments introduce their currencies via open market operations.
• In goods market households use local currency to buy the local good subject to a cash-in-advance constraint. Sell their endowment of local good for local currency.
• Can transfer cash between asset market and goods market for a fixed cost $\gamma$. Distributed over households by $f(\gamma)$. 
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Timing

Asset market

Starting bonds \( B \)

Rate of money growth \( \mu \) observed

Ending bonds \( B' \)

Asset market constraint

Bonds:
\[
B = \int qB' + P(x + \gamma) \quad \text{if cash transferred.}
\]
\[
B = \int qB' \quad \text{if no transfer.}
\]

If transfer \( x \), pay fixed cost \( \gamma \)

Cash-in-advance constraint

Consumption:
\[
c = \mu + x \quad \text{if cash transferred.}
\]
\[
c = \mu \quad \text{if no transfer.}
\]

Goods market

Starting cash \( P_{-1y} \)

Real balances \( n = P_{-1y}/P \)

Ending cash \( Py \)

Shopper

Endowment sold for cash \( Py \)

Worker
Asset Market Details

- Only uncertainty is the growth rate of money $\mu_t = \frac{M_t}{M_{t-1}}$. State $s_t = (\mu_t, \mu^*_t)$. $s_t$ is known before making decisions.
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$$B(s^t, \gamma) = \int_{s_{t+1}} q(s^t, s_{t+1}) B(s^t, s_{t+1}, \gamma) ds_{t+1} + P(s^t)[x(s^t, \gamma) + \gamma] z(s^t, \gamma)$$
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- Government’s budget constraint is given by

$$B(s^t) = M(s^t) - M(s^{t-1}) + \int_{s_{t+1}} q(s^t, s_{t+1}) B(s^t, s_{t+1}) ds_{t+1}$$
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- At time 0, there is no goods market $M(s_0) = \bar{M}$ is given and the agent comes in with an endowment of $e_0\bar{B}_h^*$ of foreign debt and $\bar{B}_h(\gamma)$ of home debt.
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- Agents have access to both asset markets but no arbitrage implies that $q(s^t, s_{t+1}) = q^*(s^t, s_{t+1}) e(s^t) / e(s^{t+1})$. WLOG home households only buy home debt.
Goods Market Details

- Households come in with real balances $n(s^t, \gamma)$.
- Consumption is then given by
  \[ c(s^t, \gamma) = n(s^t, \gamma) + x(s^t, \gamma)z(s^t, \gamma) \]  \hspace{1cm} (1)
- The real balance next period is then given by $n(s^{t+1}, \gamma) = \frac{P(s^t)y}{P(s^{t+1})}$ with the caveat that in period $t = 1$ the real balance $n(s^1, \gamma)$ is given by $M_0/P(s^1)$
- Agents have preferences
  \[ \sum_{t=1}^{\infty} \beta^t \int U(c(s^t, \gamma))g(s^t)ds^t \]
over consumption of the home good.
Equilibrium

- The resource constraint for the home market is given by

$$\int \left[ c(s^t, \gamma) + \gamma z(s^t, \gamma) \right] f(\gamma) d\gamma = y$$

(2)
Equilibrium

- The resource constraint for the home market is given by

\[ \int [c(s^t, \gamma) + \gamma z(s^t, \gamma)] f(\gamma) d\gamma = y \]  \hspace{1cm} (2)

- As fixed costs and transfers \(x(s^t, \gamma)\) are paid with cash obtained in the asset market the money market clearing condition is

\[ \int (n(s^t, \gamma) + [x(s^t, \gamma) + \gamma] z(s^t, \gamma)) f(\gamma) d\gamma = M(s^t)/P(s^t) \]  \hspace{1cm} (3)
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\]

• An *Equilibrium* is then a collection of bond and good prices \( (q, q^*) \) and \( (P, P^*) \) together with bond holdings \( (B, B^*) \) and allocations \( (c, x, z, n) \) (similarly for foreign) such that the households are maximizing, the government’s budget constraint holds, and the resource and money market clearing constraints hold.
The Planner’s Problem

• We can combine equations (1), (2) and (3) to give us that \( y = \frac{M(s^t)}{P(s^t)} \).
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- We can combine this with $n(s^t, \gamma) = \frac{P(s^t-1)y}{P(s^t)}$ to get that $n(s^t, \gamma) = y/\mu_t$ (holds at $t = 1$ with different argument). This is consumption for the inactive households.
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• In this economy inflation is only distorting because it induces
  households to use real resources to pay the fixed cost \( \gamma \). This and
  the fact that households have a complete set of nominal claims
  means that an equilibrium will solve the planners problem

\[
\max_{c_A, z} \sum_{t=1}^{\infty} \beta^t \int_{s^t} \int_{\gamma} U(c(s^t, \gamma)) f(\gamma) g(s^t) d\gamma ds^t
\]

subject to the resource constraint and

\[
c(s^t, \gamma) = z(s^t, \gamma)c_A(s^t, \gamma) + [1 - z(s^t, \gamma)]\frac{y}{\mu_t}
\]
Solving the planners problem

- The planners problem above is derived under the assumption that $\widetilde{B}(\gamma)$ and $\widetilde{B}^*(\gamma)$ are chosen such that Pareto weights are equal.
- It is clear that that the planners problem is time independent and thus it reduces to finding $c_A(\mu)$ and $\widetilde{\gamma}(\mu)$ to solve

$$\max \ U(c_A(\mu))F(\widetilde{\gamma}(\mu)) + U(y/\mu)[1 - F(\widetilde{\gamma}(\mu))]$$

subject to

$$c_A(\mu)F(\widetilde{\gamma}(\mu)) + \int_{\gamma}^{\widetilde{\gamma}(\mu)} \gamma f(\gamma)d\gamma + (y/\mu)[1 - F(\widetilde{\gamma}(\mu))] = y$$

- If $\gamma < \widetilde{\gamma}(\mu_t)$ then $c(s^t, \gamma) = c_A(\mu_t)$ and otherwise $c(s^t, \gamma) = y/\mu_t$. 
Pricing

• The active consumers are the ones who price the assets and thus the pricing kernels become (similarly for foreign kernel)

\[ m(s^t, s_{t+1}) = \beta \frac{U'(c_A(\mu_{t+1})}{U'(c_A(\mu_{t}))} \frac{1}{\mu_{t+1}} \]

• The risk free dollar return then satisfies \( R_{t+1}^{-1} = E_t m_{t+1} \) (also \( R_{t+1}^{-1} = E_t m^*_{t+1} \))

• No arbitrage \( R_{t+1} = R_{t+1}^* e_{t+1} / e_t \) then gives us that \( m_{t+1} e_{t+1} / e_t \) is a pricing kernel for the foreign good, which combined with complete markets gives us

\[ \log e_{t+1} - \log e_t = \log m^*_{t+1} - \log m_{t+1} \]

• Taking logs of our equations for the risk free rate we get \( i_t = -\log E_t m_{t+1} \) and thus our risk premium can be expressed as

\[ \rho_t = (E_t \log m^*_{t+1} - E_t \log m_{t+1}) - (\log E_t m^*_{t+1} - \log E_t m_{t+1}) \]
Quadratic Approximation

• Take the second order log approximation of $U'(c_A(\mu_t))$ around $\bar{\mu}$ to get

$$\log U'(c_A(\mu_t)) = \log U'(c_A(\bar{\mu})) - \phi \hat{\mu} + \frac{1}{2} \eta \hat{\mu}^2$$

where $\hat{\mu}_t = \log \mu_t - \log \bar{\mu}$. Under CRRA $\phi = \sigma \frac{d \log c_A(\mu)}{d \log u}|_{\mu=\bar{\mu}}$

• With this quadratic approximation we have that the pricing kernel is given by

$$\log m_{t+1} = \log \frac{\beta}{\bar{\mu}} - (\phi + 1)\hat{\mu}_{t+1} + \frac{1}{2} \eta \hat{\mu}_{t+1}^2 + \phi \hat{\mu} - \frac{1}{2} \eta \hat{\mu}^2$$

• We will assume that the log of home money growth has normal innovations so

$$\hat{\mu}_{t+1} = \mathbb{E}_t \hat{\mu}_{t+1} + \epsilon_{t+1}$$

$\epsilon_{t+1}$ has mean zero and variance $\sigma_{\epsilon}^2$
Forward Premium Anomaly

- The interest rate differential can be written as 
  \[ i_t - i_t^* = \mathbb{E}_t \log e_{t+1} - \log e_t - p_t. \]
  Letting \( \nu_t \) be the real exchange rate \( e_t P_t^*/P_t \) one obtains

  \[ \mathbb{E}_t \log e_{t+1} - \log e_t = (\mathbb{E}_t \log \nu_{t+1} - \log \nu_t) + \mathbb{E}_t [\log \mu_{t+1} - \log \mu_{t+1}^*] \]
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- As the real exchange rate is the ratio of the marginal utilities of the active agents one can show that
  \[ \frac{d}{d\hat{\mu}_t} (\mathbb{E}_t \log \nu_{t+1} - \log \nu_t) = \phi \left[ \frac{d(\mathbb{E}_t \hat{\mu}_{t+1})}{d\hat{\mu}_t} - 1 \right] \]
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- Thus if
  \[ \frac{d(\mathbb{E}_t \hat{\mu}_{t+1})}{d\hat{\mu}_t} \leq \frac{\phi}{\phi + 1} \]
  an increase in money growth will lead to an expected appreciation of the exchange rate.
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- We also know the following formula, which gives us

  \[ \frac{dp_t}{d\hat{\mu}_t} = -\frac{\eta(\phi + 1)\sigma^2}{1 - \eta \sigma^2} \frac{d(\mathbb{E}_t \hat{\mu}_{t+1})}{d\hat{\mu}_t} \]
Forward Premium Proposition

Proposition

If these inequalities are satisfied

\[
\frac{\phi(1 - \eta \sigma^2)}{1 + \phi} < \frac{d \hat{\mu}_{t+1}}{d \hat{\mu}_t} \leq \frac{\phi}{1 + \phi}
\]

then for \( \mu_t \) close to \( \bar{\mu} \), a change in money growth leads the interest rate differential and the expected exchange rate depreciation to move in opposite directions.
Numerics

• Note that planners problem can be solved without knowing the process for $\mu$.
• One can then specify

$$\hat{\mu}_{t+1} = g(\hat{\mu}_t) + \epsilon_{t+1}$$

to obtain the behavior one requires for the relationship between exchange rate and interest rate. (for instance their baseline is chosen such that the exchange rate is Martingale)
• These numerical results match the data qualitatively with exchange rates being more variable than interest rate differentials
• Endogenous segmentation is critical, as exogenous segmentation does not give necessary variation in interest rate differentials.
Advantages

• Market segmentation is a compelling mechanism for explaining the variation in the market price for risk.

• The change in consumption of active agents provides an intuitive mechanisms for the variation in discount factor and explaining the puzzles presented in the paper.

• More over the paper is able to account for these puzzles with a simple general equilibrium model.
Limitations

- Unfortunately the results are very fragile and rely on knife edge situations.
- Model relies on the cost of transferring wealth from asset to goods market being real.
- It also relies upon the initial endowment of bonds being such that the planer’s problem with equal weights constitutes a competitive equilibrium.
- It would be interesting to see if these results hold in a more generic economy, especially allowing the cost of transferring wealth to be in nominal terms.