The Term Structure of Interest Rates in a DSGE Model with Recursive Preferences

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Presentation by Matt Smith

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Goal is to generate realistic asset prices and quantities jointly

Want to discipline choice of parameters

This paper: Estimates a model with Epstein Zin preferences and production using Maximum Likelihood
Ingredients

- Epstein-Zin Preferences
- Investment Adjustment costs
- Exogenously specified inflation process
- Third order approximation
- Particle filter to evaluate the likelihood
- ML estimation
Epstein-Zin Preferences

\[ U_t = \left[ \left( c_t^\nu (1 - l_t)^{1-\nu} \right)^{\frac{1-\gamma}{\theta}} + \beta \left( E_t U_{t+1}^{1-\gamma} \right)^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}} \]

\[ \theta \equiv \frac{1-\gamma}{1-\frac{1}{\varphi}} \]

- \( \varphi \) is IES
- \( \gamma \) is Risk Aversion
- \( \beta \) is discount factor
Technology:
\[ \log(z_{t+1}) = \lambda + \log(z_t) + \sigma_z \epsilon_{z,t+1} \]

Cobb-Douglas Production
\[ y_t = k_t^\zeta (z_t l_t)^{1-\zeta} \]

Capital Evolution:
\[ k_{t+1} = (1 - \delta) k_t + G(i_t/k_t)k_t \]

Adjustment costs:
\[ G\left(\frac{i_t}{k_t}\right) = a_1 + a_2 \left(\frac{i_t}{k_t}\right)^{1-\frac{1}{\tau}} \]

Aggregate Feasibility:
\[ c_t + i_t = y_t \]

Exogenous Inflation:
\[ \log(\pi_{t+1}) = \log(\bar{\pi}) + \rho(\log(\pi_t) - \log(\bar{\pi})) + (\sigma_w \omega_{t+1} + \kappa_0 \sigma_z \epsilon_{z,t+1}) + \iota(\sigma_w \omega_t + \kappa_1 \sigma_z \epsilon_{z,t}) \]

Budget constraint:
\[ c_t + i_t + \frac{b_{t+1}}{p_t} \frac{1}{R_t} = r_t k_t + w_t l_t + \frac{b_t}{p_t} \]
Planners Problem

\[ V_t = \max\{c_t, i_t, k_{t+1}, l_t\} U_t \]

st.

\[ c_t + i_t = k_t^\zeta (z_t l_t)^{1-\zeta} \]

\[ k_{t+1} = (1 - \delta) k_t + G(\frac{i_t}{k_t}) k_t \]
Bond Pricing

SDF:

$$M_{t+1} = \beta \left( \frac{c_{t+1}^\nu (1-l_{t+1})^{1-\nu}}{c_t^\nu (1-l_t)^{1-\nu}} \right)^{\frac{1-\gamma}{\theta}} \frac{c_t}{c_{t+1}} \left( \frac{V_{t+1}^{1-\gamma}}{E_t[V_{t+1}^{1-\gamma}]} \right)^{1-\frac{1}{\theta}}$$

One Period nominal bond price $R_t^{-1}$:

$$E_t \left( M_{t+1} \frac{1}{\pi_{t+1}} \right) = \frac{1}{R_t}$$

or,

$$E_t \left( M_{t+1} \frac{1}{\pi_{t+1}} \frac{1}{R_{t+1,t+1}} \right) = \frac{1}{R_{t,t+1}}$$

Pricing recursion:

$$E_t \left( M_{t+1} \frac{1}{\pi_{t+1}} \frac{1}{R_{t+1,t+m}} \right) = \frac{1}{R_{t,t+m}}$$
Equilibrium Conditions

Quantities:

\[ V_t = \left[ \left( c_t^\nu (1 - l_t)^{1-\nu} \right)^{\frac{1-\gamma}{\theta}} + \beta \left( E_t V_{t+1}^{1-\gamma} \right)^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}} \]

\[ \left( \frac{i_t}{k_t} \right)^{\frac{1}{\tau}} = E_t \left[ M_{t+1} \left( a_2 \zeta k_{t+1}^{\xi-1} \left( z_{t+1} l_{t+1} \right)^{1-\zeta} + \left( \frac{i_{t+1}}{k_{t+1}} \right)^{\frac{1}{\tau}} \left( 1 - \delta + G \left( \frac{i_{t+1}}{k_{t+1}} \right) \right) \right) \right] \]

\[ c_t + i_t = k_t^{\xi} \left( z_t l_t \right)^{1-\zeta} \]

\[ \frac{1-\nu}{\nu} \frac{c_t}{1-l_t} = (1 - \zeta) k_t^{\xi} z_t^{1-\zeta} l_t^{-\zeta} \]

\[ k_{t+1} = (1 - \delta) k_t + G \left( \frac{i_t}{k_t} \right) k_t \]

Prices:

\[ E_t \left( M_{t+1} \frac{1}{\pi_{t+1}} \frac{1}{R_{t+1,t+m}} \right) = \frac{1}{R_{t,t+m}} \]
Perturbation in General

The equilibrium conditions can be written as:

\[ F(y_{t+1}, y_t, x_{t+1}, x_t) = 0 \]

The solution of the model is of the form:

\[ y_t = g(x_t, \chi) \]
\[ x_{t+1} = h(x_t, \chi) + \chi \epsilon_{t+1}, \]

\( \chi \) is the perturbation parameter:
\( \chi = 0 \rightarrow \) deterministic model
\( \chi = 1 \rightarrow \) model you are interested in

Perurbation idea: approximate functions \( g() \) and \( h() \) by Taylor polynomials about the point \((x_{ss}, 0)\)
Build approximation sequentially

If the $k - 1^{th}$ order derivatives are known then the $k^{th}$ order derivatives are $\tilde{X}^k$ in the linear system $A^{(k)} \tilde{X}^k = b^{(k)}$

$A^{(k)}$ and $b^{(k)}$ arise from taking the $k^{th}$ derivative of

$$F\left(\underbrace{g(h(x_t, \chi) + \chi \epsilon_{t+1}, \chi)}_{y_{t+1}}, \underbrace{g(x_t, \chi)}_{y_t}, \underbrace{h(x_t, \chi) + \chi \epsilon_{t+1}, x_t}_{x_{t+1}}\right) = 0$$

and substituting the derivatives found for orders $k - 1$ and below
Equilibrium Conditions

Denote $\tilde{var}_t = \frac{var_t}{z_{t-1}}$

Quantity equilibrium conditions

$$F(\tilde{k}_t, \log(\tilde{z}_t), \chi) = 0$$

Bond Pricing equilibrium conditions

$$\bar{F}(\tilde{k}_t, \log(\tilde{z}_t), \log(\pi_t), \omega_t, \chi) = 0$$

Want Operator:

$$V(\cdot), c(\cdot), k(\cdot), l(\cdot), i(\cdot), \{R_m(\cdot)\}^{20}_{m=1}$$

that solve the above system of equations
States deviations for quantities:

\[ s_t = (\tilde{k}_t - \tilde{k}_{ss}, \log(\tilde{z}_t) - \log(\tilde{z}_{ss}), 1) \]

3rd order approximation of variable \( \text{var} \in \{V_t, c_t, l_t, i_t, k_{t+1}\} \):

\[ \text{var}(\tilde{k}_t, \log(\tilde{z}_t), 1) \approx \text{var}_{ss} + \text{var}_{i,ss}s_t^i + \frac{1}{2}\text{var}_{ij,ss}s_t^i s_t^j + \frac{1}{6}\text{var}_{ijl,ss}s_t^i s_t^j s_t^l \]

State deviations for bond pricing:

\[ s_{at} = (\tilde{k}_t - \tilde{k}_{ss}, \log(\tilde{z}_t) - \log(\tilde{z}_{ss}), \log(\pi_t) - \log(\bar{\pi}), \omega_t, 1) \]

3rd order approximation of variable \( \text{var} \in \{R_{t,t+m}\} \):

\[ \text{var}(\tilde{k}_t, \log(\tilde{z}_t), \log(\pi_t), \omega_t, 1) \approx \text{var}_{ss} + \text{var}_{i,ss}sa_t^i + \frac{1}{2}\text{var}_{ij,ss}sa_t^i sa_t^j \\
+ \frac{1}{6}\text{var}_{ijl,ss}sa_t^i sa_t^j sa_t^l \]

Fun Fact: \( \gamma \) only appears in the 'constant' term in the second order coefficients, and 3rd order coefficients involving two derivatives wrt the perturbation parameter.
Estimating parameters in a non-linear state space

Observables: \( Y_t \)
Econometric States: \( S_t \)
Vector of parameters: \( \theta \)

\[
Y_t = g(S_t, \nu_t) \\
S_{t+1} = h(S_t, w_{t+1})
\]

Need to evaluate the likelihood:

\[
p(Y_{1:T}|\theta) = \prod_{t=1}^{T} p(Y_t|Y_{1:t-1}) = \prod_{t=1}^{T} \int p(Y_t|S_t)p(S_t|Y_{1:t-1})dS_t
\]

This paper uses a particle filter to do so, then maximize this likelihood.
If we had $N$ draws, $\{S_t^i\}_{j=1}^N$ where $S_t^i \sim p(S_t|Y_{1:t-1})$, we can estimate the likelihood by:

$$p(Y_{1:T} | \theta) = \prod_{t=1}^T \left( \frac{1}{N} \sum_{j=1}^N p(Y_t | S_t^j) \right)$$

Through the use of propagation and resampling steps, a particle filter allows us evolve a discrete approximation of $p(S_{1:t-1}|Y_{1:t-1})$ to one of $p(S_{1:t}|Y_{1:t})$, and along the way, compute the likelihood contribution $p(Y_t|Y_{1:t-1})$.
Observables

\[ Y_t = \begin{bmatrix} 
\log\left(\frac{c_t}{c_{t-1}}\right) \\
\log\left(\frac{y_t}{y_{t-1}}\right) \\
R_{t,t+4} \\
R_{t,t+8} \\
R_{t,t+12} \\
R_{t,t+16} \\
R_{t,t+20} \\
\log(\pi_t) 
\end{bmatrix} \]

Repeat exercise first omitting \( \log(\pi_t) \), then again omitting 
\( \left( \log\left(\frac{c_t}{c_{t-1}}\right), \log\left(\frac{y_t}{y_{t-1}}\right), \log(\pi_t) \right) \)
Observables

\[ Y_t = \begin{bmatrix}
\log(\tilde{c}_{ss} + c_i s_t^i + \frac{1}{2} c_{ij} s_t^i s_t^j + \frac{1}{6} c_{iij} s_t^i s_t^j s_t^k) - \log(\tilde{c}_{ss} + \tilde{c}_t - 1) + \log(\tilde{z}_{t-1}) + \lambda \\
\tilde{y}_{ss} + y_i s_t^i + \frac{1}{2} y_{ij} s_t^i s_t^j + \frac{1}{6} y_{ijkl} s_t^i s_t^j s_t^k - \log(\tilde{y}_{ss} + \tilde{y}_t - 1) + \log(\tilde{z}_{t-1}) + \lambda \\
R_{ss}^{(4)} + R_{i}^{(4)} s_t^i + \frac{1}{2} R_{ij}^{(4)} s_t^i s_t^j + \frac{1}{6} R_{ijl}^{(4)} s_t^i s_t^j s_t^k \\
R_{ss}^{(8)} + R_{i}^{(8)} s_t^i + \frac{1}{2} R_{ij}^{(8)} s_t^i s_t^j + \frac{1}{6} R_{ijl}^{(8)} s_t^i s_t^j s_t^k \\
R_{ss}^{(12)} + R_{i}^{(12)} s_t^i + \frac{1}{2} R_{ij}^{(12)} s_t^i s_t^j + \frac{1}{6} R_{ijl}^{(12)} s_t^i s_t^j s_t^k \\
R_{ss}^{(16)} + R_{i}^{(16)} s_t^i + \frac{1}{2} R_{ij}^{(16)} s_t^i s_t^j + \frac{1}{6} R_{ijl}^{(16)} s_t^i s_t^j s_t^k \\
R_{ss}^{(20)} + R_{i}^{(20)} s_t^i + \frac{1}{2} R_{ij}^{(20)} s_t^i s_t^j + \frac{1}{6} R_{ijl}^{(20)} s_t^i s_t^j s_t^k \\
\log(\tilde{\pi}) + \rho \log(\tilde{\pi}_t) + \kappa_0 \log(\tilde{z}_t) + \nu (\sigma_\omega \omega_{t-1} + \kappa_1 \sigma_z \log(\tilde{z}_{t-1}))
\end{bmatrix} + \begin{bmatrix}
\sigma_{\nu_1} \nu_{1,t} \\
\sigma_{\nu_2} \nu_{2,t} \\
\sigma_{\nu_3} \nu_{3,t} \\
\sigma_{\nu_4} \nu_{4,t} \\
\sigma_{\nu_5} \nu_{5,t} \\
\sigma_{\nu_6} \nu_{6,t} \\
\sigma_{\nu_7} \nu_{7,t} \\
\sigma_{\omega_7} \omega_{t}
\end{bmatrix}
\]
Econometric States

\[
S_{t+1} = \begin{bmatrix}
k_{t+1} \\
\log(\tilde{z}_{t+1}) \\
\log(\pi_{t+1}) \\
w_{t+1} \\
1 \\
\hat{c}_t \\
\hat{y}_t \\
\log(\pi_t) \\
\log(\tilde{z}_t) \\
w_t
\end{bmatrix} = \begin{bmatrix}
k_i s^i_t + \frac{1}{2} k_{ij} s^i_t s^j_t + \frac{1}{6} k_{iij} s^i_t s^j_t s^l_t \\
\sigma_z \epsilon_{t+1} \\
\log(\bar{\pi}) + \rho \log(\bar{\pi}_t) + \kappa_0 \log(\bar{\tilde{z}}_t) + \iota(\sigma_\omega \omega_{t-1} + \kappa_1 \sigma_z \log(\bar{\tilde{z}}_{t-1})) \\
w_{t+1} \\
1 \\
c_i s^i_t + \frac{1}{2} c_{ij} s^i_t s^j_t + \frac{1}{6} c_{iij} s^i_t s^j_t s^l_t \\
\gamma_i s^i_t + \frac{1}{2} \gamma_{ij} s^i_t s^j_t + \frac{1}{6} \gamma_{iij} s^i_t s^j_t s^l_t \\
\log(\bar{\pi}_t) \\
\log(\bar{\tilde{z}}_t) \\
w_t
\end{bmatrix}
\]
<table>
<thead>
<tr>
<th></th>
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<th></th>
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<td>1.67%</td>
<td>5.56%</td>
<td>5.76%</td>
<td>5.93%</td>
<td>6.06%</td>
<td>6.15%</td>
<td>3.43%</td>
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<td>St.dev.</td>
<td>1.96%</td>
<td>3.74%</td>
<td>2.91%</td>
<td>2.87%</td>
<td>2.80%</td>
<td>2.76%</td>
<td>2.72%</td>
<td>2.33%</td>
</tr>
<tr>
<td>25%</td>
<td>0.98%</td>
<td>-0.22%</td>
<td>3.42%</td>
<td>3.63%</td>
<td>3.84%</td>
<td>3.99%</td>
<td>4.03%</td>
<td>1.79%</td>
</tr>
<tr>
<td>50%</td>
<td>2.11%</td>
<td>1.84%</td>
<td>5.36%</td>
<td>5.45%</td>
<td>5.59%</td>
<td>5.65%</td>
<td>5.71%</td>
<td>2.76%</td>
</tr>
<tr>
<td>75%</td>
<td>3.25%</td>
<td>3.82%</td>
<td>7.15%</td>
<td>7.31%</td>
<td>7.44%</td>
<td>7.57%</td>
<td>7.67%</td>
<td>4.46%</td>
</tr>
</tbody>
</table>

Table 1: The table reports the summary statistics of consumption growth, output growth, bond yields, inflation, and hours worked. All statistics are expressed in annual terms. The sample period is 1953.Q1 to 2008.Q4.
Notes on Estimation

- Measurement error added to all observables except for inflation
- St. Dev of measurement error set so that "the model explains 75% of the standard deviation observed in the data"
- Labor not included in the observables but still influences $\nu$
- Calibrated Parameters

\begin{align*}
\zeta &= 0.3 \\
\delta &= 0.029 \\
\rho &= 0.955 \\
\bar{\pi} &= 1.009 \\
\lambda &= 0.045
\end{align*}
### Estimation Results

The Term Structure of Interest Rates in a DSGE Model with Recursive Preferences

<table>
<thead>
<tr>
<th>Data</th>
<th>Cons. gr., Output gr., Yields, Inflation</th>
<th>Cons. gr., Output gr., Yields</th>
<th>Yields</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MLE</td>
<td>Std.Error</td>
<td>MLE</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.994</td>
<td>0.0001</td>
<td>0.994</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>79.34</td>
<td>12.234</td>
<td>88.23</td>
</tr>
<tr>
<td>$\psi$</td>
<td>1.731</td>
<td>0.2124</td>
<td>2.087</td>
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<tr>
<td>$\tau$</td>
<td>0.032</td>
<td>0.0061</td>
<td>0.063</td>
</tr>
<tr>
<td>$\kappa_0$</td>
<td>-0.053</td>
<td>0.0088</td>
<td>-0.012</td>
</tr>
<tr>
<td>$\kappa_1$</td>
<td>-0.522</td>
<td>0.1018</td>
<td>-0.174</td>
</tr>
<tr>
<td>$\sigma_\epsilon$</td>
<td>0.008</td>
<td>0.0009</td>
<td>0.008</td>
</tr>
<tr>
<td>$\sigma_\omega$</td>
<td>0.002</td>
<td>0.0002</td>
<td>0.003</td>
</tr>
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Table 2: Point Estimates and Standard Errors
### Panel A: Means

<table>
<thead>
<tr>
<th></th>
<th>Cons. gr.</th>
<th>Output gr.</th>
<th>1Y</th>
<th>2Y</th>
<th>3Y</th>
<th>4Y</th>
<th>5Y</th>
<th>Inflation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observed Data</td>
<td>2.06%</td>
<td>1.67%</td>
<td>5.56%</td>
<td>5.76%</td>
<td>5.93%</td>
<td>6.06%</td>
<td>6.15%</td>
<td>3.43%</td>
</tr>
<tr>
<td>All data</td>
<td>2.12%</td>
<td>2.11%</td>
<td>5.92%</td>
<td>5.98%</td>
<td>6.05%</td>
<td>6.09%</td>
<td>6.09%</td>
<td>3.67%</td>
</tr>
<tr>
<td>All data, but no inflation</td>
<td>2.12%</td>
<td>2.11%</td>
<td>5.63%</td>
<td>5.78%</td>
<td>5.92%</td>
<td>6.04%</td>
<td>6.13%</td>
<td>3.65%</td>
</tr>
<tr>
<td>Yields</td>
<td>2.12%</td>
<td>2.10%</td>
<td>5.54%</td>
<td>5.74%</td>
<td>5.90%</td>
<td>5.99%</td>
<td>6.11%</td>
<td>3.68%</td>
</tr>
</tbody>
</table>

### Panel B: Volatilities

<table>
<thead>
<tr>
<th></th>
<th>Cons. gr.</th>
<th>Output gr.</th>
<th>1Y</th>
<th>2Y</th>
<th>3Y</th>
<th>4Y</th>
<th>5Y</th>
<th>Inflation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>1.96%</td>
<td>3.74%</td>
<td>2.91%</td>
<td>2.87%</td>
<td>2.80%</td>
<td>2.76%</td>
<td>2.72%</td>
<td>2.33%</td>
</tr>
<tr>
<td>All data</td>
<td>2.40%</td>
<td>2.91%</td>
<td>1.79%</td>
<td>1.64%</td>
<td>1.50%</td>
<td>1.38%</td>
<td>1.28%</td>
<td>2.32%</td>
</tr>
<tr>
<td>All data, but no inflation</td>
<td>2.54%</td>
<td>2.95%</td>
<td>3.28%</td>
<td>3.00%</td>
<td>2.75%</td>
<td>2.53%</td>
<td>2.33%</td>
<td>3.75%</td>
</tr>
<tr>
<td>Yields</td>
<td>2.32%</td>
<td>2.85%</td>
<td>3.31%</td>
<td>3.03%</td>
<td>2.78%</td>
<td>2.56%</td>
<td>2.46%</td>
<td>3.79%</td>
</tr>
</tbody>
</table>

Table 3: Means (Panel A) and volatilities (Panel B) of consumption growth, output growth, five bond yields, and inflation.
Table 4: Autocorrelation of consumption growth (Panel A), the 1-year bond yield (Panel B), and inflation (Panel C).