# The Term Structure of Interest Rates in a DSGE Model with Recursive Preferences

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# Asset Pricing in Production Economies

- Goal is to generate realistic asset prices and quantities jointly
- Want to discipline choice of parameters

This paper: Estimates a model with Epstein Zin preferences and production using Maximum Likelihood

# Ingredients

- Epstein-Zin Preferences
- Investment Adjustment costs
- Exogenously specified inflation process
- Third order approximation
- Particle filter to evaluate the likelihood
- ML estimation

# Epstein-Zin Preferences

$$U_t = \left[ \left( c_t^{\nu} (1 - l_t)^{1-\nu} \right)^{\frac{1-\gamma}{\theta}} + \beta \left( E_t U_{t+1}^{1-\gamma} \right)^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}}$$

$$\theta \equiv \frac{1-\gamma}{1-\frac{1}{\varphi}}$$

- $\varphi$ is IES
- $\gamma$ is Risk Aversion
- $\beta$ is discount factor

Technology:

$$log(z_{t+1}) = \lambda + log(z_t) + \sigma_z \epsilon_{z,t+1}$$

Cobb-Douglas Production

$$y_t = k_t^{\zeta} (z_t l_t)^{1-\zeta}$$

Capital Evolution:

$$k_{t+1} = (1-\delta) k_t + G(\tfrac{i_t}{k_t}) k_t$$

Adjustment costs:

$$G(\tfrac{i_t}{k_t}) = a_1 + \tfrac{a_2}{1-\tau} \left(\tfrac{i_t}{k_t}\right)^{1-\frac{1}{\tau}}$$

Aggregate Feasibility:

$$c_t + i_t = y_t$$

Exogenous Inflation:

$$\begin{aligned} log(\pi_{t+1}) = \; & log(\bar{\pi}) + \rho(log(\pi_t) - log(\bar{\pi})) + (\sigma_w \omega_{t+1} + \kappa_0 \sigma_z \epsilon_{z,t+1}) \\ & + \iota(\sigma_w \omega_t + \kappa_1 \sigma_z \epsilon_{z,t}) \end{aligned}$$

Budget constraint:

$$c_t + i_t + \tfrac{b_{t+1}}{p_t} \tfrac{1}{R_t} = r_t k_t + w_t l_t + \tfrac{b_t}{p_t}$$

# Planners Problem

$$
\begin{aligned}
V_t &= max_{\{c_t, i_t, k_{t+1}, l_t\}} U_t \\
st. & \\
c_t + i_t &= k_t^{\zeta} (z_t l_t)^{1-\zeta} \\
k_{t+1} &= (1-\delta) k_t + G(\frac{i_t}{k_t}) k_t
\end{aligned}
$$

# Bond Pricing

SDF:

$$M_{t+1} = \beta \left( \frac{c_{t+1}^{\nu} (1-l_{t+1})^{1-\nu}}{c_t^{\nu} (1-l_t)^{1-\nu}} \right)^{\frac{1-\gamma}{\theta}} \frac{c_t}{c_{t+1}} \left( \frac{V_{t+1}^{1-\gamma}}{E_t[V_{t+1}^{1-\gamma}]} \right)^{1-\frac{1}{\theta}}$$

One Period nomial bond price $R_t^{-1}$:

$$E_t \left( M_{t+1} \frac{1}{\pi_{t+1}} \right) = \frac{1}{R_t}$$

or,

$$E_t \left( M_{t+1} \frac{1}{\pi_{t+1}} \frac{1}{R_{t+1,t+1}} \right) = \frac{1}{R_{t,t+1}}$$

Pricing recursion:

$$E_t \left( M_{t+1} \frac{1}{\pi_{t+1}} \frac{1}{R_{t+1,t+m}} \right) = \frac{1}{R_{t,t+m}}$$

# Equilibrium Conditions

Quantities:

$$
\begin{aligned}
V_t &= \left[ \left( c_t^{\nu} \left(1 - l_t\right)^{1-\nu} \right)^{\frac{1-\gamma}{\theta}} + \beta \left( E_t V_{t+1}^{1-\gamma} \right)^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}} \\
\left( \frac{i_t}{k_t} \right)^{\frac{1}{\tau}} &= E_t [ M_{t+1} ( a_2 \zeta k_{t+1}^{\zeta-1} (z_{t+1} l_{t+1})^{1-\zeta} + \left( \frac{i_{t+1}}{k_{t+1}} \right)^{\frac{1}{\tau}} (1 - \delta + G \left( \frac{i_{t+1}}{k_{t+1}} \right) ) ) ] \\
c_t + i_t &= k_t^{\zeta} (z_t l_t)^{1-\zeta} \\
\frac{1-\nu}{\nu} \frac{c_t}{1-l_t} &= (1-\zeta) k_t^{\zeta} z_t^{1-\zeta} l_t^{-\zeta} \\
k_{t+1} &= (1-\delta) k_t + G\left(\frac{i_t}{k_t}\right) k_t
\end{aligned}
$$

Prices:

$$
E_t \left( M_{t+1} \frac{1}{\pi_{t+1}} \frac{1}{R_{t+1,t+m}} \right) = \frac{1}{R_{t,t+m}}
$$

# Perturbation in General

The equilibrium conditions can be written as:

$$F(y_{t+1}, y_t, x_{t+1}, x_t) = 0$$

The solution of the model is of the form:

$$\begin{aligned} y_t &= g(x_t, \chi) \\ x_{t+1} &= h(x_t, \chi) + \chi \epsilon_{t+1}, \end{aligned}$$

$\chi$ is the perturbation parameter:
$\chi = 0 \rightarrow$ deterministic model
$\chi = 1 \rightarrow$ model you are interested in
Perurbation idea: approximate functions g() and h() by Taylor polynomials about the point $(x_{ss}, 0)$

- Build approximation sequentially
- If the $k-1^{th}$ order derivatives are known then the $k^{th}$ order derivatives are $\tilde{X}^k$ in the linear system $A^{(k)}\tilde{X}^k = b^{(k)}$
- $A^{(k)}$ and $b^{(k)}$ arise from taking the $k^{th}$ derivative of

$$F(\underbrace{g(h(x_t,\chi)+\chi\epsilon_{t+1},\chi)}_{y_{t+1}},\underbrace{g(x_t,\chi)}_{y_t},\underbrace{h(x_t,\chi)+\chi\epsilon_{t+1}}_{x_{t+1}},x_t) \quad = \quad 0$$

and substituting the derivatives found for orders $k-1$ and below

# Equilibrium Conditions

Denote $\tilde{var}_t = var_t / z_{t-1}$

Quantity equilibrium conditions

$$F(\tilde{k}_t, log(\tilde{z}_t), \chi) = 0$$

Bond Pricing equilibrium conditions

$$\bar{F}(\tilde{k}_t, log(\tilde{z}_t), log(\pi_t), \omega_t, \chi) = 0$$

Want Operator:

$$V(\cdot), c(\cdot), k(\cdot), l(\cdot), i(\cdot), \{R_m(\cdot)\}_{m=1}^{20}$$

that solve the above system of equations

# Perturbation solution, $\chi = 1$

States deviations for quantities:

$$s_t = (\tilde{k}_t - \tilde{kss}, log(\tilde{z}_t) - log(\tilde{z_{ss}}), 1)$$

3rd order approximation of variable $var \in \{V_t, c_t, l_t, i_t, k_{t+1}\}$:

$$var(\tilde{k}_t, log(\tilde{z}_t), 1) \approx var_{ss} + var_{i,ss} s_t^i + \frac{1}{2} var_{ij,ss} s_t^i s_t^j + \frac{1}{6} var_{ijl,ss} s_t^i s_t^j s_t^l$$

State deviations for bond pricing:

$$sa_t = (\tilde{k}_t - \tilde{kss}, log(\tilde{z}_t) - log(\tilde{z}_{ss}), log(\pi_t) - log(\bar{\pi}), \omega_t, 1)$$

3rd order approximation of variable $var \in \{R_{t,t+m}\}$:

$$\begin{aligned} var(\tilde{k}_t, log(\tilde{z}_t), log(\pi_t), \omega_t, 1) \approx\ & var_{ss} + var_{i,ss} sa_t^i + \frac{1}{2} var_{ij,ss} sa_t^i sa_t^j \\ & + \frac{1}{6} var_{ijl,ss} sa_t^i sa_t^j sa_t^l \end{aligned}$$

Fun Fact: $\gamma$ only appears in the 'constant' term in the second order coefficients, and 3rd order coefficients involving two derivatives wrt the perturbation parameter

# Estimating parameters in a non-linear state space

Observables: $Y_t$
Econometric States: $S_t$
Vector of parameters: $\theta$

$$\begin{aligned} Y_t &= g(S_t, v_t) \\ S_{t+1} &= h(S_t, w_{t+1}) \end{aligned}$$

Need to evaluate the likelihood:

$$\begin{aligned} p(Y_{1:T}|\theta) &= \prod_{t=1}^{T} p(Y_t|Y_{1:t-1}) \\ &= \prod_{t=1}^{T} \int p(Y_t|S_t) p(S_t|Y_{1:t-1}) dS_t \end{aligned}$$

This paper uses a particle filter to do so, then maximize this likelihood.

# 1 Slide on Particle Filtering

If we we had N draws, $\{S_t^j\}_{j=1}^N$ where $S_t^j \sim p(S_t|Y_{1:t-1})$, we can estimate the likelihood by:

$$p(Y_{1:T}|\theta) = \prod_{t=1}^{T}\left(\frac{1}{N}\sum_{j=1}^{N} p(Y_t|S_t^j)\right)$$

Through the use of propogation and resampling steps, a particle filter allows us evolve a discrete approximation of $p(S_{1:t-1}|Y_{1:t-1})$ to one of $p(S_{1:t}|Y_{1:t})$, and along the way, compute the likelihood contribution $p(Y_t|Y_{1:t-1})$

# Observables

$$Y_t = \begin{bmatrix} log(\frac{c_t}{c_{t-1}}) \\ log(\frac{y_t}{y_{t-1}}) \\ R_{t,t+4} \\ R_{t,t+8} \\ R_{t,t+12} \\ R_{t,t+16} \\ R_{t,t+20} \\ log(\pi_t) \end{bmatrix}$$

Repeat exercise first omitting $log(\pi_t)$, then again omitting $(log(\frac{c_t}{c_{t-1}}), log(\frac{y_t}{y_{t-1}}), log(\pi_t))$

# Observables

$$Y_t = \begin{bmatrix} log(\tilde{c}_{ss} + c_i s_t^i + \frac{1}{2} c_{ij} s_t^i s_t^j + \frac{1}{6} c_{ijl} s_t^i s_t^j s_t^l) - log(\tilde{c}_{ss} + \tilde{c}_{t-1}) + \widehat{log(\tilde{z}_{t-1})} + \lambda \\ \tilde{y}_{ss} + y_i s_t^i + \frac{1}{2} y_{ij} s_t^i s_t^j + \frac{1}{6} y_{ijl} s_t^i s_t^j s_t^l - log(\tilde{y}_{ss} + \tilde{y}_{t-1}) + \widehat{log(\tilde{z}_{t-1})} + \lambda \\ R_{ss}^{(4)} + R_i^{(4)} sa_t^i + \frac{1}{2} R_{ij}^{(4)} sa_t^i sa_t^j + \frac{1}{6} R_{ijl}^{(4)} sa_t^i sa_t^j sa_t^l \\ R_{ss}^{(8)} + R_i^{(8)} sa_t^i + \frac{1}{2} R_{ij}^{(8)} sa_t^i sa_t^j + \frac{1}{6} R_{ijl}^{(8)} sa_t^i sa_t^j sa_t^l \\ R_{ss}^{(12)} + R_i^{(12)} sa_t^i + \frac{1}{2} R_{ij}^{(12)} sa_t^i sa_t^j + \frac{1}{6} R_{ijl}^{(12)} sa_t^i sa_t^j sa_t^l \\ R_{ss}^{(16)} + R_i^{(16)} sa_t^i + \frac{1}{2} R_{ij}^{(16)} sa_t^i sa_t^j + \frac{1}{6} R_{ijl}^{(16)} sa_t^i sa_t^j sa_t^l \\ R_{ss}^{(20)} + R_i^{(20)} sa_t^i + \frac{1}{2} R_{ij}^{(20)} sa_t^i sa_t^j + \frac{1}{6} R_{ijl}^{(20)} sa_t^i sa_t^j sa_t^l \\ log(\bar{\pi}) + \rho \widehat{log(\pi_t)} + \kappa_0 \widehat{log(\tilde{z}_t)} + \iota(\sigma_\omega \omega_{t-1} + \kappa_1 \sigma_z \widehat{log(\tilde{z}_{t-1})}) \end{bmatrix} + \begin{bmatrix} \sigma_{\nu_1} \nu_{1,t} \\ \sigma_{\nu_2} \nu_{2,t} \\ \sigma_{\nu_3} \nu_{3,t} \\ \sigma_{\nu_4} \nu_{4,t} \\ \sigma_{\nu_5} \nu_{5,t} \\ \sigma_{\nu_6} \nu_{6,t} \\ \sigma_{\nu_7} \nu_{7,t} \\ \sigma_\omega \omega_t \end{bmatrix}$$

# Econometric States

$$
S_{t+1} = \begin{bmatrix} k_{t+1} \\ \widehat{log(\tilde{z}_{t+1})} \\ \widehat{log(\pi_{t+1})} \\ \omega_{t+1} \\ 1 \\ \hat{\hat{c}}_t \\ \hat{\hat{y}}_t \\ \widehat{log(\pi_t)} \\ \widehat{log(\tilde{z}_t)} \\ \omega_t \end{bmatrix} = \begin{bmatrix} k_i s_t^i + \frac{1}{2} k_{ij} s_t^i s_t^j + \frac{1}{6} k_{ijl} s_t^i s_t^j s_t^l \\ \sigma_z \epsilon_{t+1} \\ log(\bar{\pi}) + \rho \widehat{log(\pi_t)} + \kappa_0 \widehat{log(\tilde{z}_t)} + \iota(\sigma_\omega \omega_{t-1} + \kappa_1 \sigma_z \widehat{log(\tilde{z}_{t-1})}) \\ \omega_{t+1} \\ 1 \\ c_i s_t^i + \frac{1}{2} c_{ij} s_t^i s_t^j + \frac{1}{6} c_{ijl} s_t^i s_t^j s_t^l \\ y_i s_t^i + \frac{1}{2} y_{ij} s_t^i s_t^j + \frac{1}{6} y_{ijl} s_t^i s_t^j s_t^l \\ \widehat{log(\pi_t)} \\ \widehat{log(\tilde{z}_t)} \\ \omega_t \end{bmatrix}
$$

| | Cons. gr. | Output gr. | Yields 1Y | 2Y | 3Y | 4Y | 5Y | Infl. | Hours |
|---|---|---|---|---|---|---|---|---|---|
| Mean | 2.06% | 1.67% | 5.56% | 5.76% | 5.93% | 6.06% | 6.15% | 3.43% | 49.99% |
| St.dev. | 1.96% | 3.74% | 2.91% | 2.87% | 2.80% | 2.76% | 2.72% | 2.33% | 1.12% |
| 25% | 0.98% | -0.22% | 3.42% | 3.63% | 3.84% | 3.99% | 4.03% | 1.79% | 49.36% |
| 50% | 2.11% | 1.84% | 5.36% | 5.45% | 5.59% | 5.65% | 5.71% | 2.76% | 49.99% |
| 75% | 3.25% | 3.82% | 7.15% | 7.31% | 7.44% | 7.57% | 7.67% | 4.46% | 50.79% |

Table 1: The table reports the summary statistics of consumption growth, output growth, bond yields, inflation, and hours worked. All statistics are expressed in annual terms. The sample period is 1953.Q1 to 2008.Q4.

# Notes on Estimation

- Measurement error added to all observables except for inflation
- St.Dev of measurment error set so that "the model explains 75% of the standard deviation observed in the data"
- Labor not included in the observables but still influences $\nu$
- Calibrated Parameters

$$
\begin{aligned}
\zeta &= 0.3 \\
\delta &= 0.029 \\
\rho &= 0.955 \\
\bar{\pi} &= 1.009 \\
\lambda &= 0.045
\end{aligned}
$$

# Estimation Results

| Data | Cons. gr., Output gr., Yields, Inflation | | Cons. gr., Output gr., Yields | | Yields | |
|---|---|---|---|---|---|---|
| | MLE | Std.Error | MLE | Std. Error | MLE | Std.Error |
| $\beta$ | 0.994 | 0.0001 | 0.994 | 0.0001 | 0.994 | 0.0002 |
| $\gamma$ | 79.34 | 12.234 | 88.23 | 10.157 | 96.75 | 20.125 |
| $\psi$ | 1.731 | 0.2124 | 2.087 | 0.2348 | 1.775 | 0.4614 |
| $\tau$ | 0.032 | 0.0061 | 0.063 | 0.0071 | 0.026 | 0.0125 |
| $\kappa_0$ | -0.053 | 0.0088 | -0.012 | 0.0045 | -0.055 | 0.0124 |
| $\iota$ | -0.522 | 0.1018 | -0.174 | 0.0598 | -0.175 | 0.0625 |
| $\kappa_1$ | -0.046 | 0.0093 | 0.235 | 0.124 | 0.102 | 0.0897 |
| $\sigma_\varepsilon$ | 0.008 | 0.0009 | 0.008 | 0.0008 | 0.008 | 0.0012 |
| $\sigma_\omega$ | 0.002 | 0.0002 | 0.003 | 0.0003 | 0.003 | 0.0005 |

Table 2: Point Estimates and Standard Errors

# Moments 1

Panel A: Means

| | Cons. gr. | Output gr. | Yields 1Y | 2Y | 3Y | 4Y | 5Y | Inflation |
|---|---|---|---|---|---|---|---|---|
| Observed Data | 2.06% | 1.67% | 5.56% | 5.76% | 5.93% | 6.06% | 6.15% | 3.43% |
| All data | 2.12% | 2.11% | 5.92% | 5.98% | 6.05% | 6.09% | 6.09% | 3.67% |
| All data, but no inflation | 2.12% | 2.11% | 5.63% | 5.78% | 5.92% | 6.04% | 6.13% | 3.65% |
| Yields | 2.12% | 2.10% | 5.54% | 5.74% | 5.90% | 5.99% | 6.11% | 3.68% |

Panel B: Volatilities

| | Cons. gr. | Output gr. | Yields 1Y | 2Y | 3Y | 4Y | 5Y | Inflation |
|---|---|---|---|---|---|---|---|---|
| Data | 1.96% | 3.74% | 2.91% | 2.87% | 2.80% | 2.76% | 2.72% | 2.33% |
| All data | 2.40% | 2.91% | 1.79% | 1.64% | 1.50% | 1.38% | 1.28% | 2.32% |
| All data, but no inflation | 2.54% | 2.95% | 3.28% | 3.00% | 2.75% | 2.53% | 2.33% | 3.75% |
| Yields | 2.32% | 2.85% | 3.31% | 3.03% | 2.78% | 2.56% | 2.46% | 3.79% |

Table 3: Means (Panel A) and volatilities (Panel B) of consumption growth, output growth, five bond yields, and inflation.

# Moments 2

Panel A: Consumption growth

| Lag length | 1Q | 2Q | 3Q | 4Q | 5Q | 6Q | 7Q | 8Q | 9Q | 10Q |
|---|---|---|---|---|---|---|---|---|---|---|
| Data | 0.32 | 0.17 | 0.18 | 0.09 | -0.03 | 0.06 | 0.01 | -0.151 | -0.04 | 0.01 |
| All data | 0.37 | 0.04 | -0.01 | -0.07 | -0.05 | -0.07 | -0.03 | -0.05 | -0.06 | 0.02 |
| All data, but no inflation | 0.38 | 0.03 | -0.01 | -0.07 | -0.06 | -0.07 | -0.03 | -0.05 | -0.06 | 0.02 |
| Yields | 0.48 | 0.07 | -0.01 | -0.08 | -0.06 | -0.07 | -0.05 | -0.06 | -0.06 | 0.01 |

Panel B: 1-year bond yield

| Lag length | 1Q | 2Q | 3Q | 4Q | 5Q | 6Q | 7Q | 8Q | 9Q | 10Q |
|---|---|---|---|---|---|---|---|---|---|---|
| Data | 0.95 | 0.91 | 0.87 | 0.81 | 0.76 | 0.71 | 0.67 | 0.64 | 0.61 | 0.58 |
| All data | 0.96 | 0.92 | 0.88 | 0.83 | 0.79 | 0.75 | 0.7 | 0.66 | 0.62 | 0.58 |
| All data, but no inflation | 0.97 | 0.93 | 0.89 | 0.83 | 0.8 | 0.75 | 0.71 | 0.67 | 0.63 | 0.6 |
| Yields | 0.95 | 0.91 | 0.87 | 0.82 | 0.78 | 0.74 | 0.69 | 0.65 | 0.61 | 0.57 |

Panel C: Inflation

| Lag length | 1Q | 2Q | 3Q | 4Q | 5Q | 6Q | 7Q | 8Q | 9Q | 10Q |
|---|---|---|---|---|---|---|---|---|---|---|
| Data | 0.88 | 0.83 | 0.8 | 0.77 | 0.71 | 0.67 | 0.61 | 0.59 | 0.55 | 0.54 |
| All data | 0.80 | 0.77 | 0.74 | 0.69 | 0.66 | 0.63 | 0.58 | 0.54 | 0.50 | 0.47 |
| All data, but no inflation | 0.94 | 0.90 | 0.86 | 0.81 | 0.77 | 0.73 | 0.68 | 0.64 | 0.60 | 0.56 |
| Yields | 0.94 | 0.90 | 0.86 | 0.81 | 0.77 | 0.73 | 0.68 | 0.64 | 0.60 | 0.56 |

Table 4: Autocorrelation of consumption growth (Panel A), the 1-year bond yield (Panel B), and inflation (Panel C).