Affine Term Structure Models and the Forward Premium Anomaly
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Reading Group Presentation

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Forward Premium Anomaly

- No arbitrage ⇒ $R_t = \frac{F_t}{S_t} R^*_t$
- Forward premium: $f_t - s_t = r_t - r^*_t$
- Forward premium regression:

$$s_{t+1} - s_t = a_1 + a_2 (f_t - s_t) + \text{residual}$$

- Results: US dollar vs. . . .

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<th>Currency</th>
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<td>0.0339</td>
<td>0.0213</td>
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<tr>
<td></td>
<td>(0.0027)</td>
<td>(0.847)</td>
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- Anomaly: high interest rate currencies tend to appreciate
Forward Premium Anomaly

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- Forward premium: $f_t - s_t = r_t - r^*_t$
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- Anomaly: high interest rate currencies tend to appreciate
Fama (1984)

- Attributes behavior to a time-varying risk-premium

- Decomposition: $f_t - s_t = f_t - \mathbb{E}_t s_{t+1} + \mathbb{E}_t s_{t+1} - s_t$

- Population regression coefficient:

$$a_2 = \frac{\text{cov}(q, p + q)}{\text{var}(p + q)} = \frac{\text{cov}(q, p) + \text{var}(q)}{\text{var}(p + q)}$$

- Fama’s conditions: $a_2 < 0$ requires
  1. $\text{cov}(p, q) < 0$
  2. $\text{var}(p) > \text{var}(q)$
Pricing kernels

For dollar returns: \( \mathbb{E}_t(m_{t+1} R_{t+1}) = 1 \)

For pound returns: \( \mathbb{E}_t(m^*_{t+1} R^*_{t+1}) = 1 \)

For converted returns: \( \mathbb{E}_t \left[ m_{t+1} (S_{t+1}/S_t) R^*_{t+1} \right] = 1 \)

- Pound asset and currencies both traded
  \( \Rightarrow \mathbb{E}_t(m^*_{t+1} R^*_{t+1}) = \mathbb{E}_t \left[ m_{t+1} (S_{t+1}/S_t) R^*_{t+1} \right] \)

- Complete markets \( \Rightarrow \) unique choice \( m^*_{t+1} = m_{t+1} S_{t+1}/S_t \)
  \( \Rightarrow s_{t+1} - s_t = \log m^*_{t+1} - \log m_{t+1} \)
Forward Rates and Risk Premiums

Pricing a forward contract at $t$

$$\mathbb{E}_t[m_{t+1}(F_t - S_{t+1})] = 0$$

Rearranging $\Rightarrow$

$$(F_t/S_t)\mathbb{E}_t(m_{t+1}) = \mathbb{E}_t(m_{t+1}S_{t+1}/S_t) = \mathbb{E}_t(m_{t+1}^*)$$

$\Rightarrow$ forward premium

$$f_t - s_t = \log \mathbb{E}_t m_{t+1}^* - \log \mathbb{E}_t m_{t+1}$$

$$= r_t - r_{t}^*$$

and $r_t = -\log \mathbb{E}_t m_{t+1}$
Forward Rates and Risk Premiums (2)

- Expected rate of depreciation:

\[ q_t = \mathbb{E}_t s_{t+1} - s_t = \mathbb{E}_t \log m_{t+1}^* - \mathbb{E}_t \log m_{t+1} \]

- \( \Rightarrow \) risk premium:

\[ p_t = (\log \mathbb{E}_t m_{t+1}^* - \mathbb{E}_t \log m_{t+1}^*) - (\log \mathbb{E}_t m_{t+1} - \mathbb{E}_t \log m_{t+1}) \]
Cumulant decomposition

- Expand $\log \mathbb{E}_t m_{t+1}$ in terms of cumulants of the conditional distribution of $\log m_{t+1}$

$$\log \mathbb{E}_t m_{t+1} = \sum_{j=1}^{\infty} \frac{\kappa_{jt}}{j!}$$

- Note that $\kappa_{1t} = \mu_{1t}$

$$\Rightarrow \quad \rho_t = \kappa_{-1,t}^* - \kappa_{-1,t} \quad \left( \kappa_{-1,t} = \sum_{j=2}^{\infty} \frac{\kappa_{jt}}{j!} \right)$$

- First moments affect $q$

- Higher moments affect $p$
General Affine Model

Duffie and Kan (1996)

- Vector of state variables $z$:

$$z_{t+1} = (I - \Phi)\theta + \Phi z_t + V(z_t)^{1/2} \varepsilon_{t+1}$$

where $\{\varepsilon_t\} \sim \text{NID}(0, I)$, $\Phi$ is stable with positive diagonal elements, and $V$ is diagonal with elements

$$v_i(z) = \alpha_i + \beta_i^\top z$$

- Given $z$ pricing kernel is

$$-\log m_{t+1} = \delta + \gamma^\top z_t + \lambda^\top V(z_t)^{1/2} \varepsilon_{t+1}$$
Standard Cox-Ingersoll-Ross

\[-\log m_{t+1} = (1 + \lambda_1^2/2) z_{1t} + \lambda_1 z_{1t}^{1/2} \varepsilon_{1,t+1}\]
\[-\log m_{t+1}^* = (1 + \lambda_2^2/2) z_{2t} + \lambda_2 z_{2t}^{1/2} \varepsilon_{2,t+1}\]

- Regression coefficient:
  \[a_2 = 1 + \frac{\lambda_1^2}{2}\]

- The model is inconsistent with the anomaly for any value of \(\lambda_1\)
Independent Factor Model

\[- \log m_{t+1} = \left(1 + \frac{\lambda_0^2}{2}\right)z_{0t} + (-1 + \frac{\lambda_1^2}{2})z_{1t} + \lambda_0 z_{0t}^{1/2} \varepsilon_{0t+1} + \lambda_1 z_{1t}^{1/2} \varepsilon_{1t+1} \]

\[- \log m^*_t = \left(1 + \frac{\lambda_0^2}{2}\right)z_{0t} + (-1 + \frac{\lambda_2^2}{2})z_{2t} + \lambda_0 z_{0t}^{1/2} \varepsilon_{0t+1} + \lambda_2 z_{2t}^{1/2} \varepsilon_{2t+1} \]

- Regression coefficient:

\[a_2 = 1 - \frac{\lambda_1^2}{2}\]

- Model can generate \(a_2 < 0\) with large enough \(\lambda_1\)

- However, parameter estimates \(\Rightarrow\) extreme distributional properties of \(z_1\)
Interdependent Factor Model

\[- \log m_{t+1} = (1 + \lambda_1^2/2)z_{1t} + (\gamma^* + \lambda_2^2/2)z_{2t} + \lambda_1 z_{1t}^{1/2} \varepsilon_{1t+1} + \lambda_2 z_{2t}^{1/2} \varepsilon_{2t+1} \]

\[- \log m_{t+1}^* = (\gamma^* + \lambda_2^2/2)z_{1t} + (1 + \lambda_1^2/2)z_{2t} + \lambda_2 z_{1t}^{1/2} \varepsilon_{1t+1} + \lambda_1 z_{2t}^{1/2} \varepsilon_{2t+1} \]

- Regression coefficient:

\[a_2 = 1 + \frac{\lambda_1^2 - \lambda_2^2}{2(1 - \gamma^*)} \]

- Model can generate \( a_2 < 0 \) with \( \lambda_2^2 > \lambda_1^2 \)

- Short rates:

\[r_t = z_{1t} + \gamma^* z_{2t} \]

\[r_t^* = \gamma^* z_{1t} + z_{2t} \]
Estimating the model

- Data: dollar-pound exchange rate, dollar short rate, and the dollar forward premium
- Use GMM to match
  - mean and variance of the dollar short rate
  - variance and autocorrelation of the dollar-pound forward premium
  - variance of the dollar-pound depreciation rate
  - regression coefficient $a_2$
Estimates

<table>
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<th>Parameter</th>
<th>Estimate</th>
<th>Standard Error</th>
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<tbody>
<tr>
<td>( \theta_1 )</td>
<td>0.005</td>
<td>( 4.950 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \sigma_1 )</td>
<td>0.017</td>
<td>0.004</td>
</tr>
<tr>
<td>( \phi_1 )</td>
<td>0.919</td>
<td>0.037</td>
</tr>
<tr>
<td>( \gamma^* )</td>
<td>0.331</td>
<td>0.080</td>
</tr>
<tr>
<td>( \lambda_1 )</td>
<td>(-5.541)</td>
<td>2.184</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>(-5.886)</td>
<td>2.202</td>
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\[ z_{it+1} = (1 - \phi_1)\theta_1 + \phi_1 z_{it} + \sigma_1 z_{it}^{1/2} \varepsilon_{it+1} \]

- “Price of risk” coefficients \( \lambda_i \) are huge!
- Counterfactual implications for other risk premiums
  Eg. long yields in the implied curve reach as high as 80 percent per annum
Conclusion

- Possible to parameterize some affine term structure models to fit the forward premium anomaly
- However, with the necessary parameters the models have serious shortcomings
If Exchange Rates Are Random Walks, Then Almost Everything We Say About Monetary Policy Is Wrong

- How do changes in short-term interest rates affect the economy?

- Standard DSGE: through conditional means

\[ r_t = - \log \mathbb{E}_t \left[ \beta \frac{U_{c,t+1}}{U_{c,t}} \frac{1}{\pi_{t+1}} \right] \]

- Two countries:

\[ r_t - r_t^* = q_t - p_t = \mathbb{E}_t s_{t+1} - s_t - p_t \]

- Exchange rate random walks \( \Rightarrow q_t \) constant
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