A Stochastic Overlapping Generations Economy with Inheritance

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Reading Group

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What is the contribution of this paper to the OLG model?

- Consider the wealth distribution in a model with two sources of randomness:
  - The life-span of the individual
  - The anuual income of the individual
- Prove the existence of stationary Markov equilibrium.
- The difference from Aiyagari (1994). Unbounded income shock.

The basic assumptions in this model

- A unisex model.
- Life span has a finite expectation. The dying individual gives birth to a child.
- Demographics are stationary.
- The span of economically productive life is treated in generality (parametrically).
- Inter-generational wealth transfer is specified
  - egalitarian inheritance
  - individual inheritance
- No insurance market.
- No taxes and government subsidies.

Model-demographic structure

- Time is discrete and runs $t = 0, 1, 2, 3, \cdots$
- For each $t \geq 0$, I_t is a copy of the unit interval used to parametrize the collection of agents alive at time t. Let $I_{k,t}$ be the subset of I_t corresponding to agents of age k at time t, so that

$$I_t = \cup_{k=0}^{\infty} I_{k,t}$$

- Let φ_t be a non-atomic probability measure on I_t.

$$\varphi_t(I_t) = \sum_{k=0}^{\infty} \varphi_t(I_{k,t}) = 1$$

- For fix individual $\alpha \in I_{k,t}$, the probability that α survives to the next period is η_k.
- We can construct age-processes for the collection of all agents so that these survival probabilities also corresponds to the propotions of agents who survive.

$$\varphi_{t+1}(I_{k+1,t+1}) = \eta_k \varphi_t(I_{k,t})$$

Stationary demographics

- Finite expectation of life span, Δ

$$\Delta = 1 + \eta_0 + \eta_0\eta_1 + \eta_0\eta_1\eta_2 + \cdots$$

- Let $\{v_k,\ k = 0,1,2\cdots\}$ be the stationary distribution of cohorts

$$v_{k+1} = \eta_k v_k \qquad k = 0, 1, 2\cdots$$

Using the normalization

$$\sum_{k=0}^{\infty} v_k = 1$$

we have the stationary distribution

$$v_0 = \frac{1}{\Delta}, \qquad v_{k+1} = \frac{\eta_k\eta_{k-1}\cdots\eta_0}{\Delta} \qquad \text{for } k \geq 0$$

Individual problem

- If agent α is of age 0 time t_0, his utility maximization problem is

$$\max_{\{S^\alpha_{k,t}\},\{x^\alpha_{k,t}\},\{b^\alpha_{k-1,t}\}} \sum_{k=1}^{\infty} (\eta_0\eta_1\cdots\eta_{k-1})u_k(x^\alpha_{k,t_0+k})$$

$$s.t. \quad S^\alpha_{k,t} = S^\alpha_{k-1,t-1} - b^\alpha_{k-1,t} + p_t Y^\alpha_k, \;\; x^\alpha_{k,t_0+k} = \frac{b^\alpha_{k-1,t}}{p_t}, \; k \geq 1$$

- Assumption 5.1: Only the newborn agents receive an inheritance.
- Assumption 5.2
  - For each k, u_k is strictly concave, nondecreasing, differentiable on $(0,+\infty)$, with $u_k(0) = 0$.
  - $\eta^* = \sup_k \eta_k < 1$
  - $m = \sup_k EY_k < \infty$
- Assumption 5.3
  - $\inf_x u'_0(x) > 0$
  - $u'_0(x) \geq u'_1(x) \geq u'_2(x) \geq \cdots, \forall x \in R.$

Characterization of policy functions

- Bellman equation of individual problem

$$V_k(s) = \sup_{0 \leq b \leq s} [u_k(\frac{b}{p}) + \eta_k EV_{k+1}(s - b + pY_{k+1})]$$

- Theorem 5.1
  - (2) There is a unique optimal stationary plan $\pi = \pi_p$ correponding to a bid function $c(s,k) = c(s,k;p)$ such that $0 \leq c(s,k) \leq s$ for all $s \geq 0$ and $k = 0, 1, \cdots$
- Lemma 5.1 $\lim_{s \to \infty} c(s,0) = \infty$.
- Lemma 5.2 The function $s \longmapsto s - c(s,0)$ is bounded.

Price-formation mechanism

- Price is formed endogenously as the ratio of money bid for consumption over the amount of goods produced and offered.

$$p_t = \frac{1}{Q} \sum_{k=1}^{\infty} \int_{I_{k-1,t-1}} b^{\alpha}_{k-1,t} \varphi_{t-1}(d\alpha)$$

where Q is the aggregate endowment of the economy

$$Q = \sum_{k=0}^{\infty} \int_{I_{k,t}} Y^{\alpha}_{k} \varphi_{t}(d\alpha)$$

- For a continuum of agents, Dubey, Mas Colell and Shubik (1980) were able to establish that a large class of price-formation mechanisms are equivalent.

The Egalitarian Inheritance case

- The wealth and age process $\{(S_n, \kappa_n) : n = 0,\ 1,\ 2 \cdots\}$ is a Markov chain

$$(S_{n+1}, \kappa_{n+1}) = \begin{cases} (S_n - c(S_n, \kappa_n) + pY_{\kappa_{n+1}}, \kappa_{n+1}) & \text{with probability } \eta_{\kappa_n} \\ (l, 0) & \text{with probability } 1 - \eta_{\kappa_n} \end{cases}$$

  where l is the legacy.
- The state $(l, 0)$ is a regeneration point for the chain. Let τ be the first hitting time of $(l, 0)$.
- Lemma 5.3

$$P_{(l,0)}[\tau > n] \leq (\eta^*)^n \quad \text{for all } n = 0, 1, 2 \cdots$$

  This is enough to guarantee the existence of a stationary distribution $\zeta_l(ds, k)$ and
- The characterization of the stationary distribution $\zeta_l(ds, k)$

$$\sum_k \int f(s,k)\zeta_l(ds,k) = \frac{1}{E_{(l,0)}[\tau]} E_{(l,0)} \sum_{n=0}^{\tau-1} f(S_n, \kappa_n)$$

The Individual Inheritance case

- The wealth and age process $\{(S_n, \kappa_n) : n = 0, 1, 2 \cdots\}$ is a Markov chain

$$
\begin{aligned}
& (S_{n+1}, \kappa_{n+1}) \\
= & \begin{cases} (S_n - c(S_n, \kappa_n) + pY_{\kappa_{n+1}}, \kappa_{n+1}) & \text{with probability} \quad \eta_{\kappa_n} \\ (S_n - c(S_n, \kappa_n) + pY_{\kappa_{n+1}}, 0) & \text{with probability } 1 - \eta_{\kappa_n} \end{cases}
\end{aligned}
$$

- Lemma 5.7: There exists a stationary distribution $\tilde{\zeta}$ for the Markov chain $\{(S_n, \kappa_n)\}$.
- Scatch of proof
    - $\{(S_n, \kappa_n)\alpha$ a (weak) Feller chain
    - $s - c(s, 0)$ is bounded
    - Theorem 12.1.2(i) of Meyn and Tweedie (1993)
- The result is weaker than that of Aiyagari (1994).

Definition of Stationary Markov Equilibrium

- Let $\{\pi^\alpha, \alpha \in I\}$ be an admissible collection of stationary strategies, let $p \in (0, \infty)$, and let μ_k be a measure defined on $I_{k,0}$ for every $k = 0,\ 1,\ 2 \cdots$. We say that $\{\pi^\alpha\}$, p, and $\{\mu_k\}$ form a *stationary Markov equilibrium* if, when the initial price is $p_0 = p$, and the initial wealth distribution for agents of age k is $\xi_{k,0}(\cdot, \omega) = \mu_k$ for all k and ω, and every agent $\alpha \in I$ plays π^α, the following hold:
  - (i) $p_t(\omega) = p$ and $\xi_{k,t}(\cdot, \omega) = \mu_k$ for all $k = 1,\ 2 \cdots$, and $t = 1,\ 2 \cdots$.
  - (ii) for every $\alpha \in I$, π^α is optimal among all strategies for agent α when every other agent $\beta \neq \alpha$ plays π^β.

Existence of Stationary Markov Equilibrium

- Observe that in both the Egalitarian Inheritance case and Individual Inheritance case, the stationary distribution has finite mean of wealth.
- We have
  - In the Egalitarian Inheritance case, if we pick $l \geq 0$ such that

$$v_0 l = \sum_k \int \{(1-\eta_k)(s - c(s,k) + pEY_k)\}\zeta_l(ds,k)$$

    Then

$$p = \frac{1}{Q}\sum_k \int c(s,k)\zeta_l(ds,k)$$

  - In the Individual Inheritance case

$$p = \frac{1}{Q}\sum_k \int c(s,k)\tilde{\zeta}(ds,k)$$

- In the Egalitarian Inheritance case, let $\zeta(ds, k)$ be a stationary distribution for the Markov chain $\{(S_n, \kappa_n)\}$ that balances inheritance and legacy. Let μ_k be the distribution of wealth among agents of age k under ζ.

$$\mu_k(A) = \zeta(A \times \{k\}), \;\; A \in \mathbf{B}([0, \infty))$$

Then p, $\{\pi^\alpha = \pi_p\}$, and $\{\mu_k\}$ form a *stationary Markov equilibrium*.

- In the Individual Inheritance case, let $\tilde{\zeta}(ds, k)$ be a stationary distribution for the Markov chain $\{(S_n, \kappa_n)\}$. Let $\tilde{\mu}_k$ be the distribution of wealth among agents of age k under $\tilde{\zeta}$.

$$\mu_k(A) = \tilde{\zeta}(A \times \{k\}), \;\; A \in \mathbf{B}([0, \infty))$$

Then p, $\{\pi^\alpha = \pi_p\}$, and $\{\tilde{\mu}_k\}$ form a *stationary Markov equilibrium*.

Intuition behind the construction of equilibrium

- To get the intuituition, let us look at the Individual Inheritance case
- Take the expectation of

$$S_1 = S_0 - c(S_0, \kappa_0) + pY_{\kappa_0}$$

with repect to the stationary distribution $\tilde{\zeta}(ds, k)$. Using the fact that $E(S_1) = E(S_0)$ by stationarity, we have

$$p = \frac{1}{Q} \sum_k \int c(s, k) \tilde{\zeta}(ds, k)$$

What do we learn from this paper?

- A rigorous mathematical treatment of stochastic OLG model
- A starting point to extend it to the realistic price-formation mechanism with unbouded wealth accumulation.