"The Internal Structure of Cities" by Robert Lucas and Esteban Rossi-Hansberg, E'mtca '02

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Macroeconomics Reading Group

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- Production has an externality. This externality depends on distance. Specialization?
- Under Externalities, there's no reason to believe that competitive prices are optimal.
- On the other hand, workers want to live closer to their jobs.
- The paper attempts to provided an Equilibrium Theory of the Structure Of Cities with these features.
- Goal: Define an Equilibrium. Prove Existence and Develop an Algorithm.
- My goal is to review the method to show Existence of equilibria with externalities.
- Conclusion: The model is very sensitive to small changes.

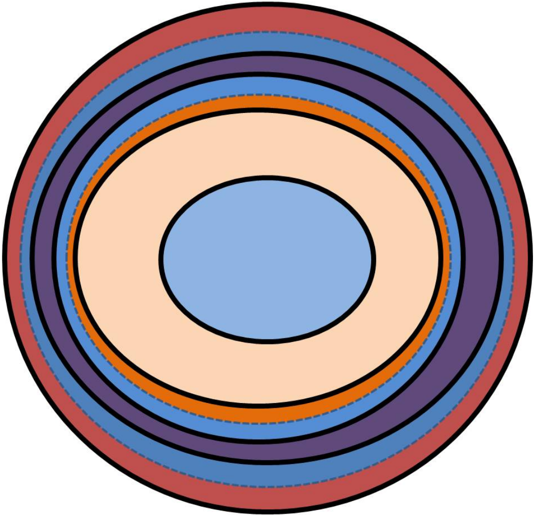

Figure: City Map

- A city is a ball of Radius S. Area πS^2. Symmetry is assumed.
- Single good is traded at unit price.
- Labor is supplies inelastically up to $\bar{u}$, which is interpreted as the opportunity cost of moving.
- Each location is described by a pair (r, ϕ).
- $\theta(r)$ is the fraction of land used for production and $1 - \theta(r)$.
- $n(r)$ is the employment density per unit of production-land.
- Total Employment at r is $2\pi\theta(r)\, n(r)$
- $l(r)$ units of land per person at r. $N(r)$ is person per unit of land and obviously $l(r) \cdot N(r) = 1$.

- Constant return to scales to land and decreasing returns to scale on labor per unit of production-land.
- Output: $x(r) = g(z(r)) f(n(r))$
- Effective Externatilty: $g(z) = z^{\gamma}$
- Effective Labor: $f(n) = An^{\alpha}$
- Distance Function: $x(r, s, \phi) = \left[r^2 - 2\cos(\phi) rs + s^2\right]$
- Externality per Location:

$$z(r) = \delta \int_0^S \int_0^{2\pi} \theta(s, \phi) n(s, \phi) s \exp(-\delta(x(r, s, \phi))) d\phi ds$$

- By symmetry, decisions are independent of angles so we can factor:

$$\psi(r, s) := \delta \int_0^{2\pi} \exp(-\delta(x(r, s, \phi))) d\phi$$

- Agents commute in rays from the circle. If leaves in s and works at r. Commuting cost is $\exp\left(-k\left|r-s\right|\right)$.
- $U\left(c,l\right)=c^{\beta}l^{1-\beta}$
- Competition must guarantee: $U\left(c\left(r\right),l\left(r\right)\right)=\bar{u}$

- An allocation are $\{(z, \theta, n, N, c, l)\}$ on $[0, S]$ such that it is determined by the above specification.
- Feasibility: Requires $\theta(r) \in [0,1], U(c(r), c(l)) = \bar{u}, l(r) \cdot N(r) = 1$ and that workers in the city live in the city and:
- **Auxiliary Stock State:**

$$H(r): \text{ stock of workers that are unhoused within the r ball.}$$

- **Auxiliary Flow:**

$$y(r) = 2\pi r [\theta(r) n(r) - (1 - \theta(r)) N(r)]$$

- **Key is the link**:

$$\begin{aligned} \frac{dH(r)}{dr} &= y(r) + \kappa H(r) \text{ if } H(r) > 0 \text{ and} \\ \frac{dH(r)}{dr} &= y(r) - \kappa H(r) \text{ if } H(r) < 0 \end{aligned}$$

- Why?
- Feasibility thus requires $H(S) \leq 0$.

- **The Firm**:

$$q(r) = g(z(r))\, f(n(r)) - w(r)\, n(r) = \max_n \{g(z(r))\, f(n) - w(r)\, n\}$$

so by Cobb-Doubglas we can back-out: $\hat{n}(w, z), \hat{q}(w, z)$

- **The Worker**:

$$\begin{aligned} w(r) &= c(r) + Q(r)\, l(r) = \min_{c,l} [c + Q(r)\, l] \\ s.t.\ U(c, l) &\geq \bar{u} \end{aligned}$$

so by Cobb-Doubglas we can back-out: $\hat{N}(w), \hat{l}(w), \hat{Q}(w), \hat{c}(w)$

- Competition for **Rents**:

$$\begin{aligned} \theta(r) &> 0 \text{ implies } q(r) \geq Q(r) \\ \theta(r) &< 1 \text{ implies } q(r) \leq Q(r) \end{aligned}$$

- No **Wage Arbitrage**:

$$\exp(-\kappa |r - s|)\, w(s) \leq w(r) \leq \exp(\kappa |r - s|)\, w(s)$$

Definition

An equilibrium is a Feasible Allocation $\{(z, \theta, n, N, c, l)\}$, such that for a given path ω, $\hat{n}, \hat{q}$and $\hat{N}, \hat{Q}$ coincide with the allocations and $H(S) = 0$.

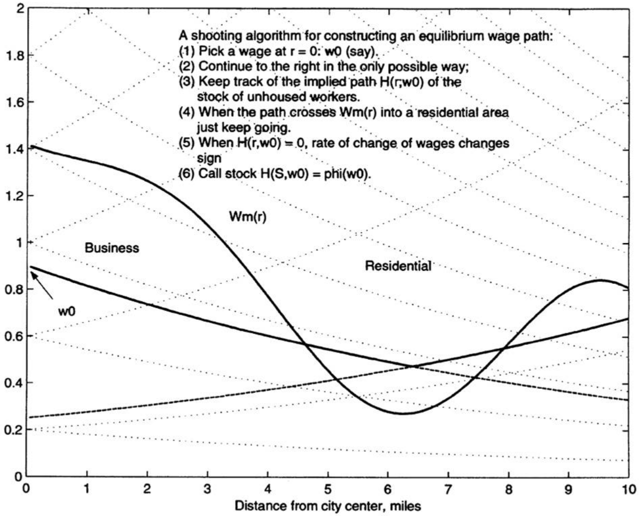

Figure: Determination of Wage Path

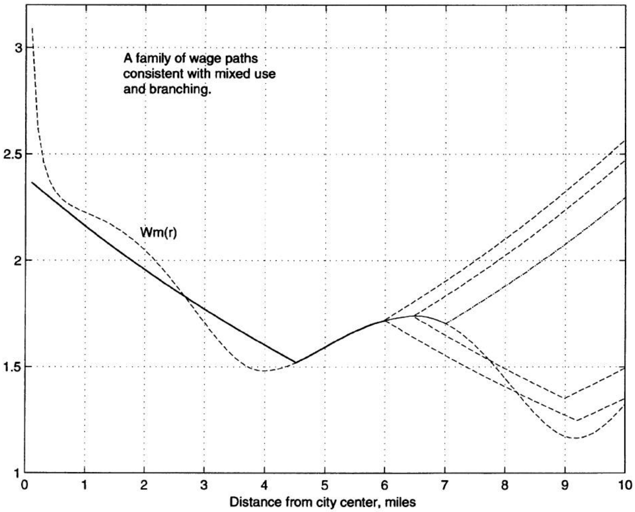

Figure: Wage Path with Separation

- Externality, the Comp. Equilibria does not Coincide Planner's.
- Technology that we know: e.g. Aiyagari..
- This is not possible here. We need another strategy. The Strategy will be as follows:
    1. -Take an externality function: $z(r)$ on $[0, S]$ and fix $w(0)$, and compute $(w, n, N, Q, q, \theta, y, H)$.
       -This objects will define a correspondence: $\varphi(\cdot; z) := \mathbb{R} \rightrightarrows \mathbb{R}$, where the image is
       $\{H(S; w(0), z) : H \text{ is computed from equilibrium in } w(0) \text{ and } z\}$
    2. Given z continuous, $\exists!, w^*$, such that $0 \in H(S; w^*, z) = 0$.**(Partial Equilibrium)**
    3. For each, z, its corresponding w^* will define a new set of decisions. Each decision will in turn define a new z' by an operator (Tz).The idea is to show that (Tz) maps $\bar{C}[0, S]$ to $\bar{C}[0, S]$. Moreover, need to show that $T(\bar{C}[0, S])$ is equi-continuous and we are able to apply Schauder's Fix Point Theorem, the generalization of Brouwer's Fixed point theorem.

Lemma

$\varphi(\cdot; z)$ is strictly decreasing: $\omega' > \omega$, implies that for $H^{'} \in \varphi(\omega'), H \in \varphi(\omega)$, then $H^{'} > H$. Every point $\varphi(w)$ is achieved by a unique path.

Lemma

$a, b \in \varphi(w)$, then $\exists c$,and $a < c < b \in \varphi(w)$.

Lemma

$\varphi(w)$ is closed.

Lemma

$\varphi(w)$ is convex.

Lemma

$\varphi(w)$ is compact and upper hemi-continuous.

Theorem

Fix z cont. There's a an allocation that is a partial equilibrium with w unique. If w coincides with $w_m^{'}$ in an interval, the w is still unique but allocation are not.

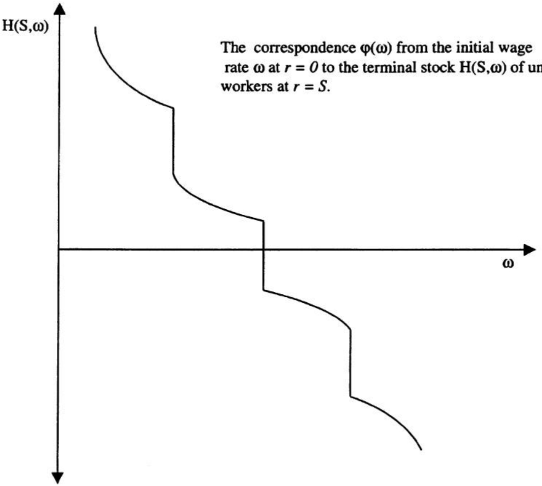

Figure: Housing Demand Residual Correspondence

Lemma
$T : C_+ \to C_+$

Lemma
Fix r*,* $w(r; z)$ *is always continuous in* z*.*

Lemma
$T : C_+ \to C_+$ *is continuous in* $\|\bullet\|$ *.*

Lemma
$w(r; z)$ *is increasing in* z*,* $(z \geq z)$

Lemma
if z *has an upper bound,* $\bar{z}$*, then* (Tz) *is also bounded by* $\bar{z}$*.*

Lemma
T *maps uniform continuous into uniform continuous.*

Lemma
Let $\bar{C}_+$ *be the set of uniformly continuous,* $T(C_+)$ *is equi-continuous.*

Theorem
Equilibria exist.

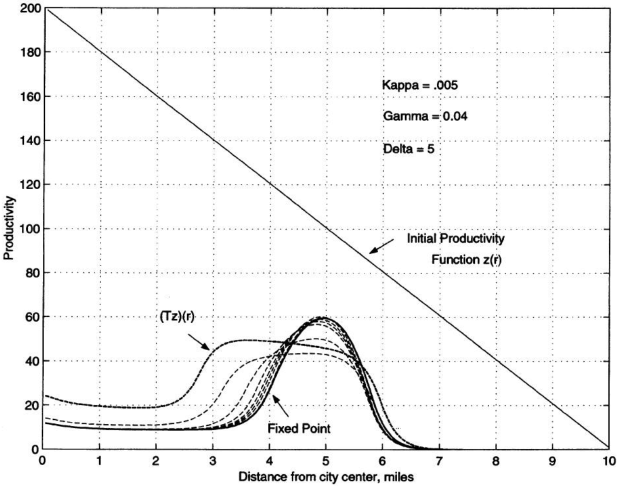

Figure: Algorithm at Work