RECURSIVE CONTRACTS

Albert Marcet
Institut d'Anàlisi Ecomòmica, CSIC
and
Ramon Marimon
European University Institute and UPF-CREi

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Abstract

We obtain a recursive formulation for a general class of contracting problems involving incentive constraints. These constraints make the corresponding maximization (*sup*) problems non recursive. Our approach consists of studying a recursive Lagrangian. Under standard general conditions, there is a recursive *saddle point* (*infsup*) functional equation (analogous to Bellman's equation) that characterizes the recursive solution for the planner's problem and the individual values. Our approach applies to a large class of dynamic contractual problems, as examples, we study the optimal policies in a model with limited enforcement and in a model with implementability constraints (as in Ramsey problems).

1 Introduction

Recursive methods have become a basic tool for the study of dynamic economic models. For example, Stokey, et al. (1989) and Ljungqvist and Sargent (2000) describe a large number of macroeconomic models that can be analyzed using recursive methods. A main advantage of such approach is that it characterizes optimal decisions -at any time t- as a time-invariant functions of a small set of state variables. However, a key condition in standard dynamic programming techniques is that only past variables can influence the set of feasible current actions. Unfortunately, many interesting economic problems do not satisfy such condition. For example, in contracting problems where agents are subject to

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intertemporal participation, or other intertemporal incentive constraints, the future development of the contract determines its feasibility. Similarly, in models of optimal policy design agent's reactions to government policies are taken as constraints and, therefore, future actions limit the set of current feasible actions available to the government.

In this paper we provide an integrated approach for a recursive formulation of a large class of dynamic models with constraints that depend on expectations of functions of future control variables. Even though these models are not recursive in the standard sense, by reformulating them as an equivalent recursive saddle point problem we obtain a recursive formulation.

We build on traditional tools of economic analysis, such as duality theory (in optimization problems), fixed point theory (in infinite dimensional spaces), and dynamic programming. We proceed in three steps. We first study the planners problem with incentive constraints (PP) as an infinite-dimensional maximization problem, for which standard duality theory applies. Second, we show the equivalence between the planner's problem and a modified saddle point problem (SPP). Third, we extend dynamic programming theory to show that the (SPP) has a recursive formulation in the sense that it satisfies a saddle point functional equation (SPFE) which generalizes Bellman's equation.

The resulting saddle point problem (SPP) expands the set of state variables to include new variables that summarize the evolution of the lagrange multipliers of the original (PP) problem. Such transformation creates some technical difficulties since the new (co)state variables can not be bounded. Fortunately, we can exploit the resulting homogeneity properties of the return function and, in this way, we are able to extend the standard contraction mapping approach to establish the relationship between SPP and the SPFE.

We show that solving the lagrangean (SPP) is equivalent to solving the recursive SPFE without concavity assumptions. This is important because incentive constraints may not have a convex structure. If concavity is satisfied, then solving the SPP (and, therefore, the SPFE) is equivalent with solving the maximization problem PP. In the absence of concavity, as in any application of lagrangean theory, our SPFE characterization is sufficient but it may not be necessary for a solution.

As standard recursive methods have proved to be useful to study many dynamic economic models -specially, but not only, in macroeconomics,- our approach has a wide range of applications. It has already proved to be useful to study models such as: growth and business cycles with possible default (Marcet and Marimon (1992), Kehoe and Perry (1998), Cooley, Marimon and Quadrini (2000)), social insurance (Attanasio and Rios-Rull (2000)) and optimal fiscal and monetary policy design with incomplete markets (Rojas (1993), Marcet, Sargent and Seppala (1996)). For brevity, however, we do not present further applications here and limit the presentation of the theory to the case of full information[1]. Our approach is related to other existing approaches that study

[1] In a follow up paper we characterize recursive contracts with incentive constraints under private information.

dynamic models with expectations constraints. In particular, to the pioneer works of Abreu, Pearce and Stacchetti (1990), Green (1987) and Thomas and Worrall (1988) and, among others, the more recent contributions of Kocherlakota (1996) and Rustichini (1998a). We briefly discuss how these, and others, works relate to ours in Section 4, after presenting the main body of the theory in Sections 2 and 3 (while most proofs are contained in the Appendix).

2 Formulating contracts as recursive saddle-point problems

In this section we briefly present our approach and develop the *saddle -point* formulation of problems with intertemporal constraints. We start by considering problems that have the following representation:

$$\textbf{PP} \qquad V(x,\, s) = \sup_{\{a_t\}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t r(x_t,\, a_t,\, s_t) \tag{1}$$

$$\text{s.t. } x_{t+1} = \ell(x_t,\, a_t,\, s_{t+1}),\;\; p(x_t, a_t,\, s_t) \geq 0, \tag{2}$$

$$\mathrm{E}_t \sum_{n=1}^{N_j+1} \beta^n h_0^j(x_{t+n},\, a_{t+n},\, s_{t+n}) + h_1^j(x_t,\, a_t,\, s_t) \geq 0, \tag{3}$$

$$j = 1, ..., l;\; t \geq 0, \quad x_0 = x,\, s_0 = s$$

$$a_t \text{ measurable with respect to } (\ldots, s_{t-1}, s_t).$$

Standard dynamic programming methods only consider constraints of the from (2) (see, for example, Stokey, *et al.* (1989) and Cooley, (1995)). However, constraints of the form (3) are not a special case of (2), since they involve expected values of future variables[2]. We know from Kydland and Prescott (1977) that, under these constraints, the usual Bellman equation is not satisfied, the solution is *not,* in general, of the form $a_t = f(x_t, s_t)$ for all t and the whole history of past shocks s_t can matter for today's optimal decision. By letting $N_j = \infty$ **PP** covers a large class of problems where discounted present values enter the implementability constraint. For example, long term contracts with *intertemporal participation constraints* take this form[3] Alternatively, by

[2]Notice that expressing (3) in the form $v(x_t, s_t) - \psi(x_t, s_t) \geq 0$, where v is the value function of **PP** and ψ some exogenously given participation constraint, is not a special case of (2) since v is not known a priori. On the other hand, combining 2) and (3) accounts for a broad class of constraints. For example, a nonlinear participation constraint of the form $g(\mathrm{E}_t \sum_{n=0}^{\infty} \beta^n h(x_{t+n},\, a_{t+n},\, s_{t+n}), x_t, a_t, s_t) \geq 0$ can easily be incorporated in our framework with one constraint of the form (2), $g(w_t, x_t, a_t, s_t) \geq 0$ (with control variables (w_t, a_t)), and a constraint of the form (3), $\mathrm{E}_t \sum_{n=0}^{\infty} \beta^n h(x_{t+n},\, a_{t+n},\, s_{t+n}) = w_t$.

[3]Examples using related methods can be found in Kocherlakota (1996) and Kletzer and Wright (1998), and using our approach in Marcet and Marimon (1992), Kehoe and Perry (1998), Attanasio and Rios-Rull (2000). Sargent and Ljungqvist (2002) provides examples with different approaches.

letting $N_j = 0$ **PP** covers problems where intertemporal reactions of agents must be taken into account. For example, dynamic Ramsey problems, where the government chooses policy variables subject to intertemporal implementability constraints, have this form[4]. We consider the two canonical cases $N_j = \infty$ and $N_j = 0$, while other intermediate cases can be easily incorporated (without loss of generality, let $N_j = \infty$, for $j = 0, ..., k$, and $N_j = 0$ for $j = k+1, ..., l$).

We consider a more general class of problems parameterized by μ,

$$\mathbf{PP}_\mu \qquad V_\mu(x,s) = \sup_{\{a_t\}} \mathrm{E}\left[\sum_{j=0}^{l}\sum_{t=0}^{N_j} \beta^t \mu^j h_0^j(x_t,\, a_t,\, s_t) \mid s\right]$$

$$\text{s.t. } x_{t+1} = \ell(x_t,\, a_t,\, s_{t+1}),\;\; p(x_t, a_t,\, s_t) \geq 0, \tag{4}$$

$$\mathrm{E}_t \sum_{n=1}^{N_j+1} \beta^n h_0^j(x_{t+n},\, a_{t+n},\, s_{t+n}) + h_1^j(x_t,\, a_t,\, s_t) \geq 0,\;\; t \geq 0 \tag{5}$$

$$x_0 = x,\; s_0 = s, \;\; N_j = \infty,\; j = 0, ..., k, N_j = 0, j = k+1, ...l$$

$$a_t \text{ measurable with respect to } (\ldots, s_{t-1}, s_t).$$

It is easy to see that **PP** is a special case of $\mathbf{PP}_\mu$. It only requires to identify the function h_0^0 with the function r, set $\mu = (1, 0, ..., 0)$ and – provided that r is bounded – choose a h_1^0 for which the corresponding participation constraint is never binding. It should be noticed that the value function, when well defined, takes the form $V_\mu(x,\, s) = \mu v_\mu(x,\, s)$.This Pareto-welfare form will play a role in some of ours characterization results. Notice that $\mathbf{PP}_\mu$ is an infinite dimensional maximization problem which, under relatively standard assumptions, has a solution in state (x, s) which is a *plan*[5] $\boldsymbol{a} \equiv \{a_t\}_{t=0}$, where $a_t(\ldots, s_{t-1}, s_t)$ is a state-contingent action (Proposition 1).

An intermediate step in our approach is to transform program $\mathbf{PP}_\mu$ into a one-period saddle-point problem $\mathbf{SPP}_\mu$ of the form[6]:

[4] Examples using the "primal approach" can be found in Chari *et al.* (1995) and Lucas and Stokey (1983); using related methods in Chang(1998) and Phelan and Stacchetti (1999), and using our approach in Rojas (1993), and previous working paper versions of this paper.

[5] We use the bold notation to denote sequences of measurable functions.

[6] We use the notation $\mu h_0(x, a, s) \equiv \sum_{j=0}^{l} \mu^j h_0^j(x, a, s)$.

$$\mathbf{SPP}_\mu \qquad \inf_{\gamma \in R^l_+} \sup_{\{a_t\}} \sum_{j=0}^{l} \left(\mu^j h_0^j(x,\, a_0,\, s) + \gamma^j h_1^j(x,\, a_0,\, s) \right)$$

$$+ \beta \mathrm{E} \left[\sum_{j=0}^{k} \left(\mu^j + \gamma^j \right) \sum_{n=1}^{\infty} \beta^t \; h_0^j(x_n,\, a_n,\, s_n) + \sum_{j=k+1}^{l} \gamma^j h_0^j(x_1,\, a_1,\, s_1) \mid s \right]$$

$$\text{s.t. } x_{t+1} = \ell(x_t,\, a_t,\, s_{t+1}), \;\; p(x_t, a_t,\, s_t) \geq 0, \quad t \geq 0$$

$$\mathrm{E}_t \sum_{n=1}^{N_j+1} \beta^n h_0^j(x_{t+n},\, a_{t+n},\, s_{t+n}) + h_1^j(x_t,\, a_t,\, s_t) \geq 0, \;\; j = 1, ..., l; \;\; t \geq 1,$$

a_t measurable with respect to $(\ldots, s_{t-1}, s_t)$.

We then show that, under fairly general conditions, solutions to $\mathbf{PP}_\mu$ are solutions to $\mathbf{SPP}_\mu$ (Theorem 1), and viceversa (Theorem 2). Advancing ideas, it should be noticed the usual slackness conditions guarantee that if $(\{a_t^*\}, \gamma^*)$ solves $\mathbf{SPP}_\mu$ in state (x, s), and $x_{t=1}^* = \ell(x_t^*,\, a_t^*,\, s_{t+1})$, $x_0^* = x$, then

$$\mathrm{E}_0 \sum_{j=0}^{l} \gamma^{j*} \left[\sum_{t=1}^{N_j+1} \beta^t h_0^j(x_t^*,\, a_t^*,\, s_t) + h_1^j(x,\, a_0^*,\, s) \right] = 0, \tag{6}$$

and therefore the value of $\mathbf{SPP}_\mu$ is $\mathrm{E}_0 \sum_{j=0}^{l} \sum_{t=0}^{N_j} \beta^t \mu^j h_0^j(x_t^*,\, a_t^*,\, s_t)$. If $\mathbf{PP}_\mu$ were a standard dynamic programming problem, then the following Bellman equation would be satisfied

$$V_\mu(x, s) = \mu h_0(x,\, a_0^*,\, s) + \beta \mathrm{E}_0 V_\mu(x^{*\prime}, s'). \tag{7}$$

The argument being that in (x, s) a $(\arg\max)$ solution $\{a_t^*\}$ to $\mathbf{SPP}_\mu$ determines a new state $(x^{*\prime}, s')$, through the Markovian transition $s \to s'$ and the state's law of motion $x^{*\prime} = \ell(x,\, a_0^*,\, s')$. However, with forward looking constraints, as in $\mathbf{PP}_\mu$, the Bellman equation (7) is *not* satisfied. A central element of our approach is to link $\mathbf{SPP}_\mu$ problems by defining a law of motion for the evolution the co-state variable μ. This link is given by the mapping φ, which takes the form: $\mu'^j = \varphi^j(\mu,\, \gamma,\, s) = \mu^j + \gamma^j$ if $N_j = \infty$, and $\mu'^j = \varphi^j(\mu,\, \gamma,\, s) = \gamma^j$ if $N_j = 0$. Then a $(\arg\min)$ solution γ^* to $\mathbf{SPP}_\mu$ in state (x, s) defines a new $\mathbf{SPP}_{\varphi(\mu,\, \gamma^*,\, s)}$ problem which can be solved in state $(x^{*\prime}, s')$.

The latter construction allows us to obtain a recursive formulation to our original $\mathbf{PP}_\mu$ problem, by showing that solutions to $\mathbf{SPP}_{\varphi(\mu,\, \gamma^*,\, s)}$ in state $(x^{*\prime}, s')$ are one period ahead solutions to $\mathbf{SPP}_\mu$ in state (x, s). More specifically, we show that, under fairly general assumptions, solutions to $\mathbf{SPP}_\mu$ obey a saddle-point functional equation **SPFE**. *A value function $W(x, \mu, s)$ satisfies the saddle-point functional equation SPFE if, and only if,*

$$\textbf{SPFE} \qquad W(x,\,\mu,\,s) = \inf_{\gamma\geq 0}\sup_{a}\{\mu h_0(x,\,a,\,s) + \gamma h_1(x,\,a,\,s) + \beta\,\mathrm{E}\,[W(x',\,\mu',\,s')|\,s]\}$$
$$\text{s.t. } x' = \ell(x,\,a,\,s),\; p(x,a,\,s) \geq 0$$
$$\text{and } \mu' = \varphi(\mu,\,\gamma,\,s).$$

To simplify the exposition we will only consider problems where policy choices are uniquely defined and, therefore, associated with a value function W satisfying **SPFE** there is *a policy function*[7] ψ; i.e. $(\,a^*,\,\gamma^*) = \psi((x,\mu,s)$.

Once we show that $V_\mu(x,s)$ satisfies the **SPFE** (Theorem 3), we show that if $W(x,\mu,s)$ satisfies **SPFE,** and its solutions are unique, the corresponding path, $\{a_t^*,\gamma_t^*\}$ generated by ψ, is a solution to $\mathbf{SPP}_\mu$ in state (x,s); that is, $(\{a_t^*\}_{t=0}^\infty,\gamma_0^*)$ solves $\mathbf{SPP}_\mu$ in state (x,s), $(\{a_t^*\}_{t=1}^\infty,\gamma_1^*)$ solves $\mathbf{SPP}_{\mu^{*\prime}}$ in state $(x^{*\prime},s')$, etc. (Theorem 4). More precisely, once we show that the value of $\mathbf{PP}_\mu$ at $(x,s), V_\mu(x,s)$, satisfies **SPFE,** we can extend the dynamic programming principle to show that the modified problem $\mathbf{PP}_\mu$ is the 'correct continuation problem' to the planners' problem, in the sense that, if μ is properly updated, the solution can be found each period by re-optimizing a version of $\mathbf{PP}_\mu$.

An implication of this recursive result is that $\mathbf{PP}_\mu$ is a way to **'solve the time-inconsistency problem,'** in the sense of defining the problem that 'the committed planner should solve' if she were given the chance to reoptimize. To see this, let $\{a_t^*\}_{t=0}^\infty$ be the solution to $\mathbf{PP}_\mu$ – for example, let $\mathbf{PP}_\mu$ be our original **PP** problem, with $\mu_0 = (1,0,...,0)$ – in state (x_0,s_0). The time-inconsistency problem arises from the fact that, if intertemporal incentive constraints of the form (3) are binding along the path leading to (x_t^*,s_t), in period t, the optimal choice from period t onwards $\left\{a_{t+j}^*\right\}_{j=0}^\infty$ *is not equal* to the series that the planner would choose if she would reoptimize $\mathbf{PP}_\mu$ in period t in state (x_t^*,s_t). In fact, $\left\{a_{t+j}^*\right\}_{j=0}^\infty$ is the solution to a properly modified planner's problem: $\mathrm{PP}_{\varphi^{(t)}(\mu_0,\,\gamma_0^*,\,s_0)}$ in state (x_t^*,s_t), where $\varphi^{(1)}(\mu_0,\,\gamma_0^*,\,s_0) \equiv \varphi(\mu_0,\,\gamma_0^*,\,s_0)$, $\varphi^{(n+1)}(\mu_0,\,\gamma_0^*,\,s_0) \equiv \varphi(\varphi^{(n)}(\mu_0,\,\gamma_0^*,\,s_0),\,\gamma_n^*,\,s_n)$, γ_0^* is the $\arg\min$ (the Lagrange multiplier) of $\mathbf{SPP}_{\mu_0}$, and γ_n^* is the $\arg\min$ of $\mathbf{SPP}_{\varphi^{(n)}(\mu_0,\,\gamma_0^*,\,s_0)}$. Only if planner were to re-optimize using $\mathrm{PP}_{\varphi^{(t)}(\mu_0,\,\gamma_0^*,\,s_0)}$ in state (x_t^*,s_t), would she follow the original plan $\{a_n^*\}_{n=t}^\infty$

In summary, when (3) only take the form of *intertemporal one-period (Euler) constraints* (i.e. $k=0$), then $\varphi^{(t)}(\mu_0,\,\gamma_0^*,\,s_0) = \gamma_t^*$. In other words, the (Ramsey) planner commits to not to react to agent's expectations or, equivalently, the (Ramsey) planner takes the 'rational expectations' of the agents as a constraint. The standard '**time-consistency problem**' takes the form of setting $\mu_{t+1} = 0$, when $\gamma_t^* > 0$[8]. Similarly, when (3) only take the form of *intertemporal participation constraints* (i.e. $k=l$), then $\varphi^{(t)}(\mu_0,\,\gamma_0^*,\,s_0) = \sum_{n=0}^t \gamma_n^*$. In other words,

[7] Our approach can be generalized to consider policy correspondences, from which measurable selections determine specific optimal choices. Marimon, Messner and Pavoni(2009) discusses this generalization.

[8] Building on these ideas, recently, Davide Debortoli and Ricardo Nunes (2007) have develop models whit mix forms of commitment.

the planner should reoptimize updating the weights in her Pareto objective function according to the aggregate shadow-value of agents' *intertemporal participation constraints* along the path. In other words, given that 'punishments to deviate are already implicit in the *intertemporal participation constraints*' the only instrument left to the planner is to relatively upgrade those agents who are tempted to default.

This interpretation of the dynamic planner's problem exploits the fact that the value function of $\mathbf{SPP}_\mu$ takes the form $W(x,\, \mu,\, s) = \mu\omega(x,\, \mu,\, s)$ and it is homogeneous of degree one in μ. In the case of *intertemporal participation constraints* $\omega_j(x,\, \mu,\, s)$ (homogeneous of degree zero in μ) corresponds to agent j's value of the contract at $(x,\, \mu,\, s)$.**SPFE,** together with (6), defines a recursive *saddle-point Bellman equation* for the planner,

$$\mu\omega(x,\, \mu,\, s) = \mu \left[h_0(x,\, a^*,\, s) + \beta\ \mathrm{E}\left[\omega(x^{*\prime},\, \mu^{*\prime},\, s')|\, s\right]\right],$$

however to have a well defined recursive problem it is necessary that a similar recursive equation is also satisfied for the individual agents. We provide conditions guaranteeing this to be the case (in Theorem 4). In particular, 'individual values ω_i are the are the subgradients W' and, therefore, they are uniquely defined if and only if $W(x,\, \mu,\, s)$ is differentiable in μ. When they are uniquely defined then it is straightforward to show that individual values are recursive.

The main results of this paper are then an immediate corollary to Theorems 1 - 4 and can be summarized as saying that, under relatively standard conditions: *i)* a $\mathbf{PP}_\mu$ problem has a recursive **SPFE** representation, and *ii)* solutions to $\mathbf{PP}_\mu$ can be obtained by solving recursive saddle point problems **SPFE**.

Theorem 4 assumes the existence of a value function W satisfying **SPFE** and a corresponding *policy function* ψ, it also makes concavity and differentiability assumptions that are not necessary for the result. The Corollary of Theorem 4 provides alternative assumptions for the same sufficiency result. To address the issue of existence of (W, ψ) and sharpen Theorem 4 we generalize dynamic programming results, to our saddle-point formulation. We first define the *Dynamic Saddle-Point Problem* as

$\mathbf{DSPP}_{(x,\mu,s)}$

$$\begin{aligned} &\inf_{\gamma \geq 0} \sup_{a} \left\{\mu h_0(x,\, a,\, s) + \gamma h_1(x,\, a,\, s) + \beta\ \mathrm{E}\left[\mu'\omega(x',\, \mu',\, s')|\, s\right]\right\} \\ &\text{s.t. } x' = \ell(x,\, a,\, s),\ p(x, a,\, s) \geq 0 \\ &\text{and } \mu' = \varphi(\mu,\, \gamma,\, s), \end{aligned}$$

we then show that, under fairly standard assumptions and for a general space of value functions M, mapping $X \times R_+^{l+1} \times S$ to R^{l+1}, $\mathbf{DSPP}_{(x,\mu,s)}$ has well defined solutions (Proposition 2). Then, $\mathbf{DSPP}_{(x,\mu,s)}$ defines then a **SPFE** operator, $T : M \longrightarrow M$, given by

$$\omega_j(x,\, \mu,\, s) = h_0^j(x,\, a^*(x,\, \mu,\, s),\, s) + \beta\ \mathrm{E}\left[\omega_j(x^{*\prime}(x,\, \mu,\, s),\, \mu^{*\prime}(x,\, \mu,\, s),\, s')|\, s\right], \tag{8}$$

if $j = 0, ..., k$, and

$$\omega_j(x,\, \mu,\, s) = h_0^j(x,\, a^*(x,\, \mu,\, s),\, s), \text{ if } j = k+1, ..., l.$$

Notice that if $a^*(x,\, \mu,\, s)$ is uniquely determined, T immediately delivers unique values ω_j, for $j = k+1, ..., l$. For $j = 0, ..., k$ we show that T is a contraction mapping. This formulation also shows a main difference between our approach and the so-called 'promise utilities approach' (see, for example, Ljungqvist and Sargent (2000), Ch. 19), where ω_j are choice variables, and the recursive equations (8) are taken as constraints. In our approach, both the individual values ω_j and their recursive structure (8) are the result of solving *Dynamic Saddle-Point Problem* in terms of pre-specified state (and co-sate) variables. Our main final theorem provides conditions under which solutions to *Dynamic Saddle-Point Problems* are solutions to our original *Planner's Problems* with intertemporal incentive (e.g. Euler-equations) or participation constraints (Theorem 5).

Before we turn to these results in Sections 4 and 5, in the next Section we show how our approach is impelemented in a couple of canonical examples.

3 Some applications

In this Section, we illustrate our approach with two examples. In the first, there are only *intertemporal participation constraints* (i.e. $k = l$ in (3)); in the second, there are only *intertemporal one-period (Euler) constraints* (i.e. $k = 0$ in (3)). The first is similar to the model studied in Marcet and Marimon (1992), Kocherlakota (1996), Kehoe and Perri (2002), among others, and it is canonical of models with intertemporal default constraints; the second is based on the model studied by Aiyagari, Marcet, Sargent and Seppala (2002) and it is canonical models with Euler constraints, as in Ramsey models.

3.1 Intertemporal participation constraints.

We consider, as an example, a model of a partnership, where several agents can share their individual risks and jointly invest in a project which can not be undertaken by single (or subgroups of) agents. Formally, there is a single good and J infinitely-lived consumers, with preferences represented by $E_0 \sum_{t=0}^{\infty} \beta^t \; u(c_{j,t})$; u is assumed to be bounded, strictly concave and monotone, with $u(0) = 0$; c represents individual consumption. Agent j receives an endowment of consumption good $y_{j,t}$ at time t and, given a realization y_t, $y_t = \sum_{j=1}^{J} y_{j,t} > 0$, agent j has an outside option $v_j^a(y_t)$. For simplicity we consider that the outside option is the autarkic solution: $v_j^a(y_t) = \mathrm{E}\left[\sum_{n=0}^{\infty} \beta^n \;\; u(y_{j,t+n}) \mid y_{j,t}\right]$, which implicitly assumes that agent j is permanently excluded from the partnership and, if she defaults, does not have any claims on the capital of the partnership[9] Total

[9] For a model where there is no permanent exclusion and, therefore, the outside option can not be taken as exogenous, see Cooley, Marimon and Quadrini (2000). Their model requires to

production is given by $F(k,\theta)$, and it can be split into consumption c and investment i. The stock of capital k depreciates at the rate δ. The joint process $\{\theta_t, y_t\}_{t=0}^{\infty}$ is assumed to be Markovian and the initial conditions (k_0, θ_0, y_0) are given. The planner's problem takes the form:

$$\textbf{PP} \qquad \max_{\{c_t, i_t\}} \; \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{j=1}^{J} \alpha_j \; u(c_{j,t})$$

$$\text{s.t. } k_{t+1} = (1-\delta)k_t + i_t, \quad F(k_t,\theta_t) + y_t - \left(\sum_{j=1}^{J} c_{j,t} + i_t \right) \geq 0$$

$$\mathrm{E}_t \sum_{n=0}^{\infty} \beta^n \; u(c_{j,t+n}) \geq v_j^a(y_t) \qquad \text{for all } j, t \geq 0.$$

It is easy to map this planner's problem into our $\mathbf{PP}_\mu$ formulation. Let $s \equiv (\theta, y)$; $x \equiv k$; $a \equiv (i, c)$; $\ell(x,\, a,\, s) \equiv (1-\delta)k + i$; $p(x,\, a,\, s) \equiv F(k,\theta) + \sum_{j\in J} y_j - \left(\sum_{j\in J} c_j + i\right)$; $h_0^0(x,\, a,\, s) \equiv \sum_{j=1}^{J} \alpha_j \; u(c_j)$, $h_0^j(x,\, a,\, s) \equiv u(c_j)$, $h_1^j(x,\, a,\, s) \equiv h_0^j(x,\, a,\, s) - v_j^a(y_t)$, $j = 1, ..., J$, and $h_1^0(x,\, a,\, s) \equiv h_0^0(x,\, a,\, s) - R$, where $R = 0$. Finally, apply the convention $\mu_0 = (1, 0, ..., 0)$ and **PP** is just $\mathbf{PP}_{\mu_0}$. In its original notation, **SPFE** takes the form

$$W(k,\, \mu, y, \theta) \; = \inf_{\gamma \geq 0} \sup_{c,i} \{ \sum_{j=1}^{J} \left((\mu_0 \alpha_j + \mu_j) \; u(c_j) - \gamma^j \left(u(c_j) - v_j^a(y) \right) \right)$$

$$+ \gamma^0 \left(\sum_{J} \alpha_j u_j(c_j) - R \right) + \beta \; \mathrm{E} \left[W(k',\, \mu', y', \theta') \,|\omega, \theta \right] \}$$

$$\text{s.t. } k' = (1-\delta)k + i, \quad F(k,\theta) + \sum_{j=1}^{J} y_j - \left(\sum_{j=1}^{J} c_j + i \right) \geq 0$$

$$\text{and } \mu' = \mu + \gamma$$

Letting ψ be the policy function associated with this functional equation, efficient allocations satisfy $(c_t,\, i_t, \gamma_t) \;=\; \psi(k_t,\, \mu_t,\, \theta_t,\, y_t)$ with initial conditions $(k_0, \overline{\mu}_0,\, \theta_0,\, y_0)$.

Notice that solutions to **SPP** satisfy $\mu_t^0 = 1$. It follows that co-state variables μ become the weights that the planner assigns to each agent, which evolve according to whether or not the participation constraint is binding. Furthermore, if u is differentiable,

$$\frac{u'(c_{i,t})}{u'(c_{j,t})} \;=\; \frac{\alpha_j + \mu_{t+1}^j}{\alpha_i + \mu_{t+1}^i} \quad , \text{ for all } i, j \text{ and } t.$$

map outside options, taken as exogenous functions, into value functions of the corresponding recursive contracts, and then find the appropriate fixed point of that map.

Thus, the optimal allocations amount to choosing efficiently the time profile of the time-dependent weights $(\alpha_j + \mu_{j,t+1})$, in such a way that the participation constraints are satisfied. Every time that the participation constraint for an agent is binding, his weight is increased by the amount of the corresponding lagrange multiplier. An agent is induced not to default by increasing his consumption not only in the period where he is tempted to default, but also for many of the following periods; in this way, the additional consumption that the agent receives to prevent default is smoothed over time. That is, individual paths of consumption depend on individual histories (in particular, on past "temptations to default") not just on the initial wealth distribution and the aggregate consumption path, as in the Arrow-Debreu competitive allocations. This also shows that if enforcement constraints are never binding (e.g., punishments are severe enough) then $\mu_t = \mu_0$ and we recover the "constancy of the marginal utility of expenditure", and the "constant proportionality between individual consumptions," given by $u'(c_{i,t})/u'(c_{j,t}) = \alpha_j / \alpha_i$. In other words, the evolution of the co-state variables can be also interpreted as the evolution of the distribution of wealth.

The intertemporal Euler equation of $\mathbf{SPP}_\mu$ is also very informative:

$$\mu^i_{t+1} u'(c_{i,t}) = \beta \mathrm{E} \left[\mu^i_{t+2} u'(c_{i,t+1}) \left(F_{k_{t+1}} + (1-\delta) \right) \right]$$

where, as with $j = 0, ..., k$, constraints: $\mu^i_{t+2} = \mu^i_{t+1} + \gamma^i_{t+1}$. That is, the 'stochastic discount factor', $\beta u'(c_{i,t+1})/u'(c_{i,t})$ is distorted by $\left(1 + \gamma^i_{t+1}/\mu^i_{t+1}\right)$, a distortion which does not vanish unless the non-negative process $\left\{\gamma^i_t\right\}$ converges to zero. But if intetemporal participation constraints are infinitely often binding, there will be a non degenerate distribution of consumption in the long-run; in contrast with an economy where intetemporal participation constraints cease to be binding, as in an economy with full enforcement.

3.2 Intertemporal implementability (Euler equation) constraints

Example 2. A Ramsey problem

We briefly sketch a version of the optimal taxation problem studied by Aiyagari, Marcet, Sargent and Seppala (2002) [to be explained better]. A representative consumer solves

$$\max E_0 \sum_{t=0}^{\infty} \delta^t \quad [u(c_t) + v(l_t)]$$

$$\text{s.t.} \qquad c_t + b_{t+1} p^b_t = l_t (1 - \tau_t) + b_t$$

The government must finance exogenous random expenditures g issuing debt and collecting taxes. Given the technology $c_t + g_t = l_t$,the budget of the government mirrors the budget of the representative agent and is subject to a no-Ponzi constraint. In a competitive equilibrium, the following intertemporal and intratemporal equations must be satisfied:

$$\begin{aligned} p_t^b u'(c_t) &= \beta\ E_t u'(c_{t+1}) \\ -\frac{v'(l_t)}{u'(c_t)} &= (1-\tau_t). \end{aligned}$$

Therefore, the Ramsey problem can be formulated as

$$\max_{\{c_t, b_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t\ [u(c_t) + v(l_t)]$$

$$\text{s.t.} \qquad c_t\ + b_{t+1}\beta\ E_t \frac{u'(c_{t+1})}{u'(c_t)} = -l_t \frac{v'(l_t)}{u'(c_t)} + b_t$$

This problem can be represented as a $\mathbf{PP}_{\mu_0}$ by letting: $s \equiv g$; $x \equiv b$; $a \equiv (c, y)$; $p(x,\ a,\ s) \equiv l - (c+g)$, $h_0^0(x,\ a,\ s) \equiv u(c_t) + v(l_t)$, $h_0^1(x,\ a,\ s) \equiv u'(c_{t+1})$, $h_1^0(x,\ a,\ s) \equiv h_0^0(x,\ a,\ s) - R$ (big); $h_1^1(x,\ a,\ s) \equiv y$, where

$$y_t = \frac{-l_t v'(l_t) + b_t u'(c_t) - c_t u'(c_t)}{b_{t+1}}\ .$$

In particular, $\mathbf{SPP}_\mu$ solutions satisfy

$$\mathrm{E}\left[(\mu_{t+1}^1 - \gamma_{t+1}^1) u'(c_{t+1})\right] = 0 \tag{9}$$

where, as with $j = k+1, ..., l$, constraints, $\mu_{t+1}^1 = \gamma_t^1$. As it can be seen, (9) immediately shows the nature of the distortion – that is, of the 'time-inconsistency' problem – and of its possible resolution: the convergence of the random variable γ_t^1. In particular, Aiyagari et al. show that, in fact, the process $\left\{\gamma_t^1\right\}$ is a non-negative submartingale.

4 The relationship between $\mathbf{PP}_\mu$, $\mathbf{SPP}_\mu$, and SPFE

This section makes more precise the relationships between the initial maximization problem $\mathbf{PP}_\mu$, the saddle-point problem $\mathbf{SPP}_\mu$ and the saddle-point functional equation **SPFE**, presented in the previous Sections. We start by introducing assumptions, which are relatively standard in convex optimization problems. We then present the main results, making explicit when some of the special assumptions. listed below, are needed for each of them. Most proofs are in the Appendix.

4.1 Assumptions and existence of solutions to $\mathbf{PP}_\mu$

Regarding the initial problem $\mathbf{PP}_\mu$, we consider the following set of assumptions:

A1. S is a countable set of an Euclidean space. The stochastic process $\{s_t\}$, $s_t \in S$, is a stationary Markovian process on the probability space $(S, \mathcal{S}, P)$.

A2. X and A are convex subsets of R^n and R^m respectively. The functions $p : X \times A \times S \rightarrow R$ and $\ell : X \times A \times S \rightarrow X$ are continuous and measurable, with respect to $(\mathcal{R}^n, \mathcal{R}^m, \mathcal{S})$. For any $(x, s) \in X \times S$ there exist $\widetilde{a} \in A$, such that $p(x, \widetilde{a}, s) > 0$.

A3. The function $\ell(\cdot, \cdot, s)$ is linear and the function $p(\cdot, \cdot, s)$ is concave.

A4. Given (x, s),there exist constants $B > 0$ and $\varphi \in (0, \beta^{-1})$, such that if $p(x, a, s) \geq 0$ and $x' = \ell(x, a, s')$, then $\|a\| \leq B \|x\|$ and $\|x'\| \leq \varphi \|x\|$

A5. The functions $h_i^j(\cdot, \cdot, s)$, $i = 0, 1, j = 0, ..., l$, are continuous and uniformly bounded, $h_0^j(x, \cdot, s)$ is non-decreasing, and $\beta \in (0, 1)$.

A5d. The functions $h_i^j(\cdot, \cdot, s)$, $i = 0, 1, j = 0, ..., l$, are continuously differentiable on $\{(x, a) : p(x, a, s) > 0\}$.

A6. The functions $h_i^j(\cdot, \cdot, s), i = 0, 1, j = 0, ..., l$, are concave.

A6s. In addition to **A6**, the functions $h_0^j(x, \cdot, s), j = 0, ..., l$, are strictly concave.

A7. For all (x, s), there exists a program $\{\widetilde{a}_n\}_{n=0}^{\infty}$, with initial conditions (x, s), which satisfies the inequality constraints (4) and (5) with strict inequality.

We take Assumptions **A1-A5** as our basic assumptions, since most dynamic equilibrium models satisfy them, and treat the concavity and inferiority assumptions, **A6-A6b** and **A7,** as special since they are not satisfied in some interesting models. It should be noticed, however, that many of our results do not rely on, simultaneously satisfying **A1-A5.** For example, that the state space is countable – instead of continuous, in **A1** – only plays a role in proving existence of solutions to $\mathbf{PP}_\mu$ (Proposition 1)[10] **.** Assumptions **A2, A3** and **A5** are standard, even if our boundedness assumption **A5** is not satisfied in some interesting examples. As in standard dynamic programming, it is possible to extend our results to unbounded returns, but we do not pursue such generalization here. Assumption **A4** allows for technologies with long-run growth. As it will made clear, **A6** is not needed for some of our main (sufficiency) results. Finally, assumption **A7** is a standard interiority assumption, only needed to guarantee the existence of Lagrange multipliers.

It is convenient to express $\mathbf{PP}_\mu$ and $\mathbf{SPP}_\mu$ as infinite dimensional maximization and saddle-point problems, respectively. For this we need to define first the space in which allocations are defined. Let $\mathcal{L}_\infty^m(S_\infty, \mathcal{S}_t, P)$ denote the space of m-valued –essentially bounded– $\mathcal{S}_t$-measurable functions. Plans are elements of $\mathcal{A} = \{\boldsymbol{a} : \forall t \geq 0,\, a_t \in \mathcal{L}_\infty^m(S_\infty, \mathcal{S}_t, P)\}$ and endogenous state variables are elements of $\mathcal{X} = \{\boldsymbol{x} : \forall t \geq 0,\, x_t \in \mathcal{L}_\infty^n(S_\infty, \mathcal{S}_t, P)\}$. Given initial conditions, (x, s), and a plan $\boldsymbol{a} \in \mathcal{A}$, we can define $\boldsymbol{x} \in \mathcal{X}$ recursively by $x_{n+1} = \ell(x_n, a_n, s_n)$,

[10]Similarly, Proposition 1 does not require a Markovian structure, although this assumption is used to obtain the general recursive structure.

where $x_0 = x$ and $s_0 = s$. It follows that the corresponding evaluation of such plan is given by

$$f_{(x,\mu.s)}(\boldsymbol{a}) \; = \mathrm{E}_0 \sum_{j=0}^{k} \sum_{t=0}^{N_j} \beta^t \mu^j h_0^j(x_t,\, a_t,\, s_t)$$

Similarly, we can describe the constraint sets by defining $g : \mathcal{A} \to \mathcal{L}_\infty^{k+1}$ and $q : \mathcal{A} \to \mathcal{L}_\infty^1$ coordinatewise as

$$\begin{aligned} g(\boldsymbol{a})\,^j_t \; &= \mathrm{E}_t \left[\sum_{n=1}^{N_j+1} \beta^n h_0^j(x_{t+n},\, a_{t+n},\, s_{t+n}) \right] + h_1^j(x_t,\, a_t,\, s_t) \\ q(\boldsymbol{a})_t \; &= p(x_t, a_t,\, s_t) \end{aligned}$$

Given initial conditions (x, s), the corresponding constraint set is then

$$\mathcal{B}(x, s) = \{\boldsymbol{a} \in \mathcal{A} : q(\boldsymbol{a})_t \geq 0,\; g(\boldsymbol{a})\,_t \;\geq 0,\; \boldsymbol{x} \in \mathcal{X},\; x_0(s) = x,\; x_{t+1} = \ell(x_t,\, a_t,\, s_t);\; t \geq 0\}$$

In summary, $\mathbf{PP}_\mu$ can be written in compact form as

$$\mathbf{PP}_\mu \qquad V_\mu(x, s) = \sup_{a \in \mathcal{B}(x,s)} f_{(x,\mu.s)}(\boldsymbol{a})$$

Proposition 1. Assume **A1-A6** and fix $\mu \in R_+^{l+1}$. There exists a program $\boldsymbol{a}^*$ which solves $\mathbf{PP}_\mu$ with initial conditions (x_0, s_0), achieving the value $V_\mu(x, s)$. Furthermore, if **A6s** is also satisfied then the solution is unique.

Proof: See Appendix.

4.2 The relationship between $\mathbf{PP}_\mu$ and $\mathbf{SPP}_\mu$

The following result follows from the standard theory of constrained optimization in linear vector spaces (see, for example, Luenberger (1969, Section 8.3, Theorem 1 and Corollary 1). Notice that, as in standard constrained optimization theory, convexity and concavity assumptions (**A2, A3,** and **A6**), as well as an interiority assumption (**A7**) are necessary in order to obtain the result.

Theorem 1 ($\mathbf{PP}_\mu \Longrightarrow \mathbf{SPP}_\mu$). Assume **A1- A6** and **A7** and fix $\mu \in R_+^{l+1}$. Let $\boldsymbol{a}^*$ be a solution to $\mathbf{PP}_\mu$ with initial conditions (x, s).There exist a $\gamma^* \in R_+^l$ such that $(\boldsymbol{a}^*, \gamma^*)$ is a solution to $\mathbf{SPP}_\mu$ in state (x, s), and the value of this latter problem is $V_\mu(x, s)$.

Proof: See Appendix.

We can also write $\mathbf{SPP}_\mu$ in a compact form, by defining

$$\mathcal{B}'(x, s) = \{\boldsymbol{a} \in \mathcal{A} : q(\boldsymbol{a})_t \geq 0 \;\&\; g(\boldsymbol{a})\,_{t+1} \;\geq 0;\; \boldsymbol{x} \in \mathcal{X},\; x_0(s) = x,\; x_{t+1} = \ell(x_t,\, a_t,\, s_t),\; t \geq 0\}$$

$$\mathbf{SPP}_\mu \qquad \inf_{\gamma \in R_+^l} \sup_{a \in \mathcal{B}'(x,s)} \left\{ f_{(x,\mu,.s)}(\boldsymbol{a}) + \gamma g(\boldsymbol{a})\,_0 \right\}$$

Theorem 2 ($\mathbf{SPP}_\mu \Longrightarrow \mathbf{PP}_\mu$). Given initial conditions $(x,\, s)$ and $\mu \in R_+^{k+1}$, let $(\boldsymbol{a}^*,\, \gamma^*)$ be a solution to $\mathbf{SPP}_\mu$. Then $\boldsymbol{a}^*$ is a solution to $\mathbf{PP}_\mu$ in state $(x,\, s)$.

Notice that, Theorem 2 is a sufficiency theorem 'almost free of assumptions.' In fact, only the basic structure of **A1- A5,** defining the corresponding infinite-dimensional optimization and saddle-point problems, is needed, *together with* assuming that a well defined solution to $\mathbf{SPP}_\mu$ exists[11]. Once these conditions are satisfied, assumptions, such as concavity (**A3**) or boundedness (**A5**), can be dispensed with.

Proof: The proof is an adaptation, to $\mathbf{SPP}_\mu$, of a sufficiency theorem for Lagrangian saddle points (see, for example, Luenberger (1969), Theorem 8.4.2, p.221). Let $(\boldsymbol{a}^*,\, \gamma^*)$ be a solution to $\mathbf{SPP}_\mu$, then let

$$\widehat{V}_\mu(x,s) = f_{(x,\mu,s)}(\boldsymbol{a}^*) + \gamma^* g(\boldsymbol{a}^*)_0.$$

Minimality of γ^* implies that, for every $\gamma \geq 0$,

$$(\gamma^* + \gamma)\, g(\boldsymbol{a}^*)_0 \geq \gamma^* g(\boldsymbol{a}^*)_0;$$

therefore, $g(\boldsymbol{a}^*)_0 \geq 0$, which together with the fact that $\boldsymbol{a}^* \in \mathcal{B}'(x,s)$, implies $\boldsymbol{a}^* \in \mathcal{B}(x,s)$; i.e. $\boldsymbol{a}^*$ is a feasible program for $\mathbf{PP}_\mu$. Furthermore, the minimality of γ^* also implies that

$$\gamma^* g(\boldsymbol{a}^*)_0 \leq 0 g(\boldsymbol{a}^*)_0 = 0$$

but since $\gamma^* \geq 0$ and $g(\boldsymbol{a}^*)_0 \geq 0$, it follows that $\gamma^* g(\boldsymbol{a}^*)_0 = 0$. Now, suppose there exist $\tilde{\boldsymbol{a}} \in \mathcal{B}(x,s)$ satisfying $f_{(x,\mu,s)}(\tilde{\boldsymbol{a}}) > f_{(x,\mu,s)}(\boldsymbol{a}^*)$, then, since $\gamma^* g(\tilde{\boldsymbol{a}})_0 \geq 0$, it must be that

$$f_{(x,\mu,s)}(\tilde{\boldsymbol{a}}) + \gamma^* g(\tilde{\boldsymbol{a}})_0 > f_{(x,\mu,s)}(\boldsymbol{a}^*) + \gamma^* g(\boldsymbol{a}^*)_0$$

which contradicts the maximality of $\boldsymbol{a}^*$ for $\mathbf{SPP}_\mu$. As a result, $\widehat{V}_\mu(x,s) = f_{(x,\mu,s)}(\boldsymbol{a}^*) + \gamma^* g(\boldsymbol{a}^*)_0 = V_\mu(x,s)$ ■

4.3 The relationship between $\mathbf{SPP}_\mu$, and SPFE

Recall that a value function W satisfies **SPFE** at $(x,\, \mu,\, s)$ if, and only if,

$$W(x,\, \mu,\, s) = \min_{\gamma \geq 0} \max_a \{\mu h_0(x,\, a,\, s) + \gamma h_1(x,\, a,\, s) + \beta\, \mathrm{E}\,[W(x',\, \mu',\, s') |\, s]\} \tag{10}$$

$$\text{s.t. } x' = \ell(x,\, a,\, s),\ p(x,a,\, s) \geq 0$$
$$\text{and } \mu' = \varphi(\mu,\, \gamma,\, s).$$

[11] In fact, to simplify the exposition, we implicitly assume that the solution is unique, although the generalization is straightforward, except when one needs to recursively connect $\mathbf{SPP}_\mu$ problems (Marimon, Messner and Pavoni (2008)).

We simply say that W satisfies **SPFE** if, and only, if it satisfies **SPFE** at *any possible state* $(x,\, \mu,\, s)$.

The *saddle-point policy correspondence* (*SP policy correspondence*) is defined by

$$
\begin{aligned}
\Psi(x,\, \mu,\, s) \quad = & \\
& \{(\, a^*, \gamma^*) : W(x, \mu, s) = \mu h_0(x,\, a^*,\, s) + \gamma^* h_1(x,\, a^*,\, s) + \beta \,\mathrm{E}\left[W(x^{*\prime},\, \mu^{*\prime},\, s')|\, s\right], \\
& \text{and} x^{*\prime} = \ell(x,\, a^*,\, s),\; p(x, a^*,\, s) \geq 0, \text{ and } \mu^{*\prime} = \varphi(\mu,\, \gamma^*,\, s)\}
\end{aligned}
$$

If Ψ is single valued, we denote it by ψ, and we call it a *saddle-point policy function* (*SP policy function*).

Theorem 3 ($\mathbf{SPP}_\mu \Longrightarrow$ SPFE). Let $W(x, \mu, s) \equiv V_\mu(x, s)$ be the value of $\mathbf{SPP}_\mu$ at (x, s), for an arbitrary $(x,\, \mu,\, s)$, then the value function W satisfies **SPFE**. In particular, if $(\boldsymbol{a}^*,\, \gamma^*)$ is a solution to $\mathbf{SPP}_\mu$ at (x, s), $(a_0^*, \gamma^*) \in \Psi(x,\, \mu,\, s)$.

As in Theorem 2, Theorem 3 is also a theorem 'almost free of assumptions,' once the underlying structure and the existence of a well defined solution to $\mathbf{SPP}_\mu$ at (x, s) is assumed.

Proof: We first proof the recursively condition (10). Let $(\boldsymbol{a}^*,\, \gamma^*)$ be a solution to $\mathbf{SPP}_\mu$ at (x, s),

$$
\begin{aligned}
W(x, \mu, s) \;&=\; f_{(x,\mu,s)}(\boldsymbol{a}^*) + \gamma^* g(\boldsymbol{a}^*)_0 \\
&=\; \mu h_0(x,\, a_0^*,\, s) + \gamma^* h_1(x,\, a_0^*,\, s) \\
&\quad + \beta \mathrm{E}\left[\sum_{j=0}^{k} \sum_{t=1}^{\infty} \beta^{t-1} \left(\mu^{*j} + \gamma^{*j}\right) h_0^j(x_t^*,\, a_t^*,\, s_t)|\, s\right] \\
&\quad + \beta \mathrm{E}\left[\sum_{j=k+1}^{l} \gamma^{*j} h_0^j(x_1^*,\, a_1^*,\, s_1)|\, s\right] \\
&=\; \mu h_0(x,\, a_0^*,\, s) + \gamma^* h_1(x,\, a_0^*,\, s) \\
&\quad + \beta \mathrm{E}\left[f_{(x_1^*, \varphi(\mu,\, \gamma^*, s), s_1)}(\boldsymbol{a}^*)|\, s\right] \\
&\leq\; \mu h_0(x,\, a_0^*,\, s) + \gamma^* h_1(x,\, a_0^*,\, s) \\
&\quad + \beta \mathrm{E}\left[V_{\varphi(\mu,\, \gamma^*, s)}(x_1^*, s_1)|\, s\right] \\
&=\; \mu h_0(x,\, a_0^*,\, s) + \gamma^* h_1(x,\, a_0^*,\, s) \\
&\quad + \beta \mathrm{E}\left[W(x_1^*, \varphi\left(\mu,\, \gamma^*, s\right), s_1)|\, s\right],
\end{aligned}
$$

where the first two equalities follow from the definition of W and f , the weak inequality follows from the fact that $V_{\varphi(\mu,\, \gamma^*, s)}(x_1^*, s_1)$ is the value of $\mathbf{PP}_{\varphi(\mu,\, \gamma^*, s)}$ at (x_1^*, s_1) and the last equality follows from Theorem 2.

To show the reverse weak inequality, it is convenient to explicitly denote by $(\boldsymbol{a}^*(x, \mu, s), \gamma^*(x, \mu, s))$ a solution to $\mathbf{SPP}_\mu$ at (x, s) and by $\boldsymbol{a}^{**}(x, \mu, s)$ a solution to $\mathbf{PP}_{\mu}$ at (x,s) (not recalling Theorem 1 for the moment), and to define the shift operator $\sigma : S^{t+1} \to S^t$ by: $\sigma(s^t) = (s_1, s_2 ..., s_t)$, where $s^t = (s_0, s_1, ..., s_t)$. We construct a sequence a^+ that consists of the optimal choice for $\mathbf{SPP}_{\mu}$ at (x,s) in the initial period, but subsequently is followed by the optimal choices for $\mathbf{PP}_{\varphi(\mu,\, \gamma^*(x,\mu,s),s)}$ at $(\ell(x,\, a_0^*(x,\mu,s),\, s_1), s_1)$. Formally a^+ is defined by

$$\begin{aligned} a_0^+(x,\mu,s) &= a_0^*(x,\mu,s) \\ a_t^+(x,\mu,s)(s^t) &= a_{t-1}^{**}(\ell(x,\, a_0^*(x,\mu,s),\, s_1), \varphi\left(\mu,\, \gamma^*(x,\mu,s), s\right),\ s_1)(\sigma(s^t)) \end{aligned}$$

for all (x,μ,s), all $t \geq 1$ and all $s^t \in S^{t+1}$. In what follows, we simplify again notation by denoting $a_t^*(x,\mu,s)$ by a_t^*, $\gamma^*(x,\mu,s)$ by γ^*, and $a_t^+(x,\mu,s)(s^t)$ by a_t^+; then, we have:

$$\begin{aligned} W(x,\mu,s) &= f_{(x,\mu,s)}(\boldsymbol{a}^*) + \gamma^* g(\boldsymbol{a}^*)_0 \\ &= \mu h_0(x,\, a_0^*,\, s) + \gamma^* h_1(x,\, a_0^*,\, s) \\ &\quad +\beta \mathrm{E}\left[\sum_{j=0}^{k}\sum_{t=1}^{\infty} \beta^{t-1} \ \left(\mu^{*j} + \gamma^{*j}\right) h_0^j(x_t^*,\, a_t^*,\, s_t) |\, s\right] \\ &\quad +\beta \mathrm{E}\left[\sum_{j=k+1}^{l} \gamma^{*j} h_0^j(x_1^*,\, a_1^*,\, s_1) |\, s\right] \\ &\geq \mu h_0(x,\, a_0^+,\, s) + \gamma^* h_1(x,\, a_0^+,\, s) \\ &\quad +\beta \mathrm{E}\left[\sum_{j=0}^{k}\sum_{t=1}^{\infty} \beta^{t-1} \ \left(\mu^{*j} + \gamma^{*j}\right) h_0^j(x_t^+,\, a_t^+,\, s_t) |\, s\right] \\ &\quad +\beta \mathrm{E}\left[\sum_{j=k+1}^{l} \gamma^{*j} h_0^j(x_1^+,\, a_1^+,\, s_1) |\, s\right] \\ &= \mu h_0(x,\, a_0^*,\, s) + \gamma^* h_1(x,\, a_0^*,\, s) \\ &\quad +\beta \mathrm{E}\left[W(x_1^*, \varphi\left(\mu,\, \gamma^*, s\right), s_1) |\, s\right], \end{aligned}$$

where the first equality follow from the definition of W and f , the weak inequality follows from the fact that $\mathbf{a}^+(x,\mu,s)$ is a feasible allocation – from (x,s) – but $\mathbf{a}^*(x,\mu,s)$ is a solution to $\mathbf{SPP}_{\mu}$ at (x,s), and the last equality follows from Theorem 1.

From the above two weak inequalities it follows that $\mathbf{SPP}_{\mu}$ values satisfy the recursive 'min max *Bellman equation*',(10). To show that the saddle-point condition of **SPFE** is also satisfied is relatively straightforward once we show that **SPFE** values are unique and we take into account that $\mathbf{SPP}_{\mu}$ already satisfies a saddle-point condition. To see uniqueness of values, consider two solutions to **SPFE** at $(x,\, \mu,\, s)$, $(\widetilde{a}, \widetilde{\gamma})$ and $(\widehat{a}, \widehat{\gamma})$, then repeated application of the saddle-point condition implies:

$$\begin{aligned}
& \mu h_0(x,\, \widetilde{a},\, s) + \widetilde{\gamma} h_1(x,\, \widetilde{a},\, s) + \beta \mathrm{E}\left[W(\ell(x,\, \widetilde{a},\, s'), \varphi(\mu,\, \widetilde{\gamma},\, s),\, s')|\, s\right] \\
\geq\ & \mu h_0(x,\, \widehat{a},\, s) + \widetilde{\gamma} h_1(x,\, \widehat{a},\, s) + \beta \mathrm{E}\left[W(\ell(x,\, \widehat{a},\, s'), \varphi(\mu,\, \widetilde{\gamma},\, s),\, s')|\, s\right] \\
\geq\ & \mu h_0(x,\, \widehat{a},\, s) + \widehat{\gamma} h_1(x,\, \widehat{a},\, s) + \beta \mathrm{E}\left[W(\ell(x,\, \widehat{a},\, s'), \varphi(\mu,\, \widehat{\gamma},\, s),\, s')|\, s\right] \\
\geq\ & \mu h_0(x,\, \widetilde{a},\, s) + \widehat{\gamma} h_1(x,\, \widetilde{a},\, s) + \beta \mathrm{E}\left[W(\ell(x,\, \widetilde{a},\, s'), \varphi(\mu,\, \widehat{\gamma},\, s),\, s')|\, s\right] \\
\geq\ & \mu h_0(x,\, \widetilde{a},\, s) + \widetilde{\gamma} h_1(x,\, \widetilde{a},\, s) + \beta \mathrm{E}\left[W(\ell(x,\, \widetilde{a},\, s'), \varphi(\mu,\, \widetilde{\gamma},\, s),\, s')|\, s\right]
\end{aligned}$$

Now, suppose that while $(\boldsymbol{a}^*,\, \gamma^*)$ is a solution to $\mathbf{SPP}_\mu$ at (x, s), there exist $\widetilde{a} \in A$, $p(x, \widetilde{a}, s) \geq 0$, such that

$$\begin{aligned}
& \mu h_0(x,\, \widetilde{a},\, s) + \gamma^* h_1(x,\, \widetilde{a},\, s) + \beta \mathrm{E}\left[W(\ell(x,\, \widetilde{a},\, s'),\, \varphi(\mu,\, \gamma^*,\, s),\;\, s')|\, s\right] \\
>\ & \mu h_0(x,\, a_0^*,\, s) + \gamma^* h_1(x,\, a_0^*,\, s) + \beta\ \mathrm{E}\left[W(\ell(x,\, a^*,\, s'),\, \varphi(\mu,\, \gamma^*,\, s),\;\, s')|\, s\right]
\end{aligned}$$

but, letting $\widetilde{a}_t^* \equiv a_t^{**}(\ell(x,\, \widetilde{a},\, s'),\, \varphi(\mu,\, \gamma^*,\, s),\;\, s')$, the first term of the inequality can be expressed in terms of the program $\widetilde{\boldsymbol{a}}^*$ as:

$$\begin{aligned}
& \mu h_0(x,\, \widetilde{a},\, s) + \gamma^* h_1(x,\, \widetilde{a},\, s) + \beta \mathrm{E}\left[W(\ell(x,\, \widetilde{a},\, s'),\, \varphi(\mu,\, \gamma^*,\, s),\;\, s')|\, s\right] \\
=\ & \mu h_0(x,\, \widetilde{a}_0,\, s) + \gamma^* h_1(x,\, \widetilde{a}_0,\, s) \\
& + \beta \mathrm{E}\left[\left(\sum_{j=0}^{k}\sum_{t=1}^{\infty} \beta^{t-1}\ \left(\mu^{*j} + \gamma^{*j}\right) h_0^j(\widetilde{x}_t^*,\, \widetilde{a}_t^*,\, s_t) + \sum_{j=k+1}^{l} \gamma^{*j} h_0^j(\widetilde{x}_1^*,\, \widetilde{a}_1^*,\, s_1)\right) |\, s\right] \\
=\ & f_{(x,\mu,s)}(\widetilde{\boldsymbol{a}}^*) +\, \gamma^* g(\widetilde{\boldsymbol{a}}^*)_0,
\end{aligned}$$

while, by (10), the second term of the inequality is simply

$$\begin{aligned}
& \mu h_0(x,\, a_0^*,\, s) + \gamma^* h_1(x,\, a_0^*,\, s) + \beta\ \mathrm{E}\left[W(\ell(x,\, a^*,\, s'),\, \varphi(\mu,\, \gamma^*,\, s),\;\, s')|\, s\right] \\
=\ & f_{(x,\mu,s)}(\boldsymbol{a}^*) +\, \gamma^* g(\boldsymbol{a}^*)_0,
\end{aligned}$$

therefore the inequality contradicts the fact that $(\boldsymbol{a}^*,\, \gamma^*)$ is a solution to $\mathbf{SPP}_\mu$ at (x, s).

Similarly, suppose there exist $\widehat{\gamma} \in R_+^l$ such that

$$\begin{aligned}
& \mu h_0(x,\, a_0^*,\, s) + \widehat{\gamma} h_1(x,\, a_0^*,\, s) + \beta\ \mathrm{E}\left[W(\ell(x,\, a^*,\, s'),\, \varphi(\mu,\, \widehat{\gamma},\, s),\;\, s')|\, s\right] \\
<\ & \mu h_0(x,\, a_0^*,\, s) + \gamma^* h_1(x,\, a_0^*,\, s) + \beta\ \mathrm{E}\left[W(\ell(x,\, a^*,\, s'),\, \varphi(\mu,\, \gamma^*,\, s),\;\, s')|\, s\right],
\end{aligned}$$

but, letting $\widehat{a}_t^* \equiv a_t^{**}(\ell(x,\, a^*,\, s'),\, \varphi(\mu,\; \widehat{\gamma},\, s),\;\, s')$, we obtain the following contradiction with the previous inequality:

$$\begin{aligned}
& \mu h_0(x,\, a_0^*,\, s) + \widehat{\gamma} h_1(x,\, a_0^*,\, s) + \beta \; \mathrm{E}\left[W(\ell(x,\, a^*,\, s'),\, \varphi(\mu,\, \widehat{\gamma},\, s),\;\, s') \vert \, s \right] \\
= \;& \mu h_0(x,\, a_0^*,\, s) + \widehat{\gamma} h_1(x,\, a_0^*,\, s) \\
& + \beta \mathrm{E}\left[\left(\sum_{j=0}^{k} \sum_{t=1}^{\infty} \beta^{t-1} \; \left(\mu^{*j} + \widehat{\gamma}^j \right) h_0^j(\widehat{x}_t^*,\, \widehat{a}_t^*,\, s_t) + \sum_{j=k+1}^{l} \widehat{\gamma}^j h_0^j(\widehat{x}_1^*,\, \widehat{a}_1^*,\, s_1) \right) \vert \, s \right] \\
\geq \;& \mu h_0(x,\, a_0^*,\, s) + \widehat{\gamma} h_1(x,\, a_0^*,\, s) \\
& + \beta \mathrm{E}\left[\left(\sum_{j=0}^{k} \sum_{t=1}^{\infty} \beta^{t-1} \; \left(\mu^{*j} + \widehat{\gamma}^j \right) h_0^j(x_t^*,\, a_t^*,\, s_t) + \sum_{j=k+1}^{l} \widehat{\gamma}^j h_0^j(x_1^*,\, a_1^*,\, s_1) \right) \vert \, s \right] \\
= \;& f_{(x,\mu,s)}(\boldsymbol{a}^*) + \widehat{\gamma} g(\boldsymbol{a}^*)_0 \\
\geq \;& f_{(x,\mu,s)}(\boldsymbol{a}^*) + \gamma^* g(\boldsymbol{a}^*)_0 \\
= \;& \mu h_0(x,\, a_0^*,\, s) + \gamma^* h_1(x,\, a_0^*,\, s) + \beta \; \mathrm{E}\left[W(\ell(x,\, a^*,\, s'),\, \varphi(\mu,\, \gamma^*,\, s),\;\, s') \vert \, s \right],
\end{aligned}$$

where the first equality follows from the definition of $\widehat{\boldsymbol{a}}^*$ and Theorem 2, the first inequality follows from the fact that $\widehat{\boldsymbol{a}}^*$ is a solution to $\mathbf{PP}\varphi(\mu,\, \widehat{\gamma},\, s)$ at $(\ell(x,\, a^*,\, s'), s')$, for every s' (following s), the next equality is definitional, the last inequality follows from the fact that $(\boldsymbol{a}^*,\, \gamma^*)$ is a solution to $\mathbf{SPP}_\mu$ at (x, s), and the last equality follows from (10) ■

The value function of $\mathbf{SPP}_\mu$ at (x, s) satisfies

$$\begin{aligned}
W(x, \mu, s) \;&=\; f_{(x,\mu,s)}(\boldsymbol{a}^*) + \gamma^* g(\boldsymbol{a}^*)_0 \\
&=\; f_{(x,\mu,s)}(\boldsymbol{a}^*) \\
&=\; \mathrm{E}_0 \sum_{j=0}^{l} \sum_{t=0}^{N_j} \beta^t \mu^j h_0^j(x_t^*,\, a_t^*,\, s_t) \\
&\equiv\; \sum_{j=0}^{l} \mu^j \omega_j(x, \mu, s) \\
&=\; \mu\omega(x, \mu, s)
\end{aligned}$$

where, for $j = 0, ...k, \omega_j(x, \mu, s) \equiv \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t h_0^j(x_t^*,\, a_t^*,\, s_t)$, and, for $j = k + 1, ...l, \omega_j(x, \mu, s) \equiv h_0^j(x_0^*,\, a_0^*,\, s_0)$. Similarly, the value function of $\mathbf{SPP}_{\varphi(\mu,\, \gamma^*,\, s)}$ at (x_1^*, s_1), $x_1^* = \ell(x,\, a_0^*,\, s)$, satisfies

$$W(x_1^*,\, \varphi(\mu,\, \gamma^*, s),\, s_1) \equiv \varphi(\mu,\, \gamma^*, s)\omega(x_1^*,\, \varphi(\mu,\, \gamma^*, s),\, s_1).$$

This representation not only has an interesting economic meaning – for example, as a 'social welfare function,' with varying weights, in problems with intertemporal participation constraints – but is also very convenient analytically. In particular, the following Corollary to Theorem 3 shows that it satisfies what we call the *saddle-point inequality property* **SPI** (Lemma 1 below shows its equivalence, for W, with **SPFE**).

A function $W(x,\mu,s)=\sum_{j=0}^{l}\mu^j\omega_j(x,\mu,s)$ satisfies the *saddle-point inequality property* **SPI** at $(x,\,\mu,\,s)$ if and only if there exist (a^*,γ^*) satisfying

$$\mu h_0(x,\,a^*,\,s)+\widetilde{\gamma}h_1(x,\,a^*,\,s)+\beta\,\mathrm{E}\left[\varphi(\mu,\,\widetilde{\gamma},s)\omega(x^{*\prime},\,\varphi(\mu,\,\gamma^*,s),\,s')|\,s\right]$$

$$\geq \mu h_0(x,\,a^*,\,s)+\gamma^*h_1(x,\,a^*,\,s)+\beta\,\mathrm{E}\left[\varphi(\mu,\,\gamma^*,s)\omega(x^{*\prime},\,\varphi(\mu,\,\gamma^*,s),\,s')|\,s\right]. \tag{11}$$

$$\geq \mu h_0(x,\,\widetilde{a},\,s)+\gamma^*h_1(x,\,\widetilde{a},\,s)+\beta\,\mathrm{E}\left[\varphi(\mu,\,\gamma^*,s)\omega(\widetilde{x}',\,\varphi(\mu,\,\gamma^*,s),\,s')|\,s\right], \tag{12}$$

for any $\widetilde{\gamma}\in R_+^{l+1}$ and $(\widetilde{a},\widetilde{x}')$ satisfying the technological constraints at $(x,\ s)$.

Corollary 3.1. ($\mathbf{SPP}_\mu \Longrightarrow$SPI). Let $W(x,\mu,s)\equiv V_\mu(x,s)$ be the value of $\mathbf{SPP}_\mu$ at (x,s), for an arbitrary $(x,\,\mu,\,s)$, then $W(x,\mu,s)=\sum_{j=0}^{l}\mu^j\omega_j(x,\mu,s)$ satisfies **SPI**.

Proof: We only need to show that (11) is satisfied, but this is immediate from the following identities

$$\begin{aligned} f_{(x,\mu,s)}(\boldsymbol{a}^*) &= \mu h_0(x,\,a_0^*,\,s)+\beta\,\mathrm{E}\left[\sum_{j=0}^{k}\mu^j\omega_j(x_1^*,\,\varphi(\mu,\,\gamma^*,s),\,s_1)|\,s\right] \\ \gamma g(\boldsymbol{a}^*)_0 &= \gamma\left[h_1(x,\,a_0^*,\,s)+\beta\,\mathrm{E}\left[\omega(x_1^*,\,\varphi(\mu,\,\gamma^*,s),\,s_1)|\,s\right]\right], \end{aligned}$$

and the definition of $\mathbf{SPP}_\mu$ at (x,s); that is, for any $\widetilde{\gamma}\in R_+^{l+1}$,

$$\begin{aligned} & \mu h_0(x,\,a^*,\,s)+\widetilde{\gamma}h_1(x,\,a^*,\,s)+\beta\,\mathrm{E}\left[\varphi(\mu,\,\widetilde{\gamma},s)\omega(x^{*\prime},\,\varphi(\mu,\,\gamma^*,s),\,s')|\,s\right] \\ = & f_{(x,\mu,s)}(\boldsymbol{a}^*)+\gamma g(\boldsymbol{a}^*)_0 \\ \geq & f_{(x,\mu,s)}(\boldsymbol{a}^*)+\gamma^* g(\boldsymbol{a}^*)_0 \\ = & \mu h_0(x,\,a^*,\,s)+\gamma^*h_1(x,\,a^*,\,s)+\beta\,\mathrm{E}\left[\varphi(\mu,\,\gamma^*,s)\omega(x^{*\prime},\,\varphi(\mu,\,\gamma^*,s),\,s')|\,s\right] \end{aligned}$$

■

The argument used in the proof of Theorem 3 can be iterated a finite number of times to show the underlying recursive structure of the $\mathbf{PP}_\mu$ formulation. If $\mathbf{PP}_\mu$ has a unique solution $\{a_t^*\}_{t=0}^{\infty}$ at (x,s), then by Theorem 1 there is a $\mathbf{SPP}_\mu$ at (x,s) with solution $(\{a_t^*\}_{t=0}^{\infty},\gamma^*)$, which in turn defines a $\mathbf{PP}_{\varphi(\mu,\,\gamma^*,\,s)}$ problem. As it has been seen in the proof of Theorem 3, $\{a_t^*\}_{t=1}^{\infty}$ solves $\mathbf{PP}_{\varphi(\mu,\,\gamma^*,\,s)}$ at $(\ell(x,\,a_0^*,\,s),s_1)$ and by Theorem 1 there is a γ_1^* such that $(\{a_t^*\}_{t=1}^{\infty},\gamma_1^*)$ solves $\mathbf{SPP}_{\varphi(\mu,\,\gamma^*,\,s)}$ at $(\ell(x,\,a_0^*,\,s),s_1)$. In turn, $\{a_t^*\}_{t=2}^{\infty}$ solves $\mathbf{PP}_{\varphi^{(2)}(\mu,\,\gamma^*,\,s)}$ at $(\ell^{(2)}(x,\,a_0^*,\,s),s_1)$ where $\varphi^{(2)}(\mu,\,\gamma^*,\,s)\equiv\varphi(\varphi(\mu,\,\gamma^*,\,s),\,\gamma_1^*,\,s_1)$ and $\ell^{(2)}(x,\,a_0^*,\,s)\equiv\ell(\ell(x,\,a_0^*,\,s),\,a_1^*,\,s_1)$. Similarly, let $\varphi^{(n+1)}(\mu,\,\gamma^*,\,s)\equiv\varphi(\varphi^{(n)}(\mu,\,\gamma^*,\,s),\,\gamma_n^*,\,s_n)$, then by recursively applying the argument of the proof of Theorem 3 we obtain the following result.

Corollary 3.2. (Recursivity of $\mathbf{PP}_\mu$). If $\mathbf{PP}_\mu$ satisfies the assumptions of Theorem 1 and has a unique solution $\{a_t^*\}_{t=0}^{\infty}$ at (x,s), then, for any (t,x_t^*,s_t), $\left\{a_{t+j}^*\right\}_{j=0}^{\infty}$ is the solution to $\mathbf{PP}_{\varphi^{(t)}(\mu,\,\gamma^*,\,s)}$ at (x_t^*,s_t), where γ^* is the minimizer of $\mathbf{SPP}_\mu$ at (x,s).

We now show that, under fairly general conditions, programs satisfying **SPFE** are solutions to $\mathbf{SPP}_\mu$ at (x, s). More formally,

Theorem 4 (SPFE $\Longrightarrow$ $\mathbf{SPP}_\mu$) Assume W, satisfying **SPFE**, is continuous in (x, μ), concave in x, and convex and homogeneous of degree one in μ**,** and assume **A5d**. Then if $(\mathbf{a}^*, \boldsymbol{\gamma}^*)$ is generated by the *SP policy function* ψ, associated with W, from an initial condition (x, μ, s) and, for all (t, s_t), $p(x_t^*, a_t^*, s_t) > 0$, then $(\mathbf{a}^*, \boldsymbol{\gamma}^*)$ is also a solution to $\mathbf{SPP}_\mu$ at (x, s).

Notice that the assumptions on W are fairly general. They are also relatively standard; in particular, if $W(x, \mu, s)$ is the value function of $\mathbf{SPP}_\mu$ at (x, s) (i.e. $W(x, \mu, s) \equiv V_\mu(x, s)$) then – as Lemma A2 in the Appendix shows – it is convex and homogeneous of degree one in μ, is continuous and bounded in (x, μ) if **A2** and **A5** are satisfied. Concavity and differentiability of W in x – which are satisfied if **A6** and **A5d** are, respectively, satisfied – are somewhat stringent assumptions. The Corollary to Theorem 4 shows how the same result **(SPFE $\Longrightarrow$ $\mathbf{SPP}_\mu$)** can be satisfied even without these assumptions. The only remaining 'stringent condition' is that $(\mathbf{a}^*, \boldsymbol{\gamma}^*)$ must be generated by a *SP policy function* ψ; i.e. must be uniquely determined[12].

Before we prove Theorem 4, we first show some interesting properties of W in Lemmas 1 and 2. These lemmas follows from several properties of convex and homogeneous functions, together with the fact that W is a value function with unique saddle-point solutions.

To simplify the exposition of these properties let $F : R_+^m \to R$ continuous, convex and homogeneous of degree one. The *subgradient set* of F at y, denoted $\partial F(y)$, is given by

$$\partial F(y) = \left\{ z \in R^m \mid F(y') \geq F(y) + (y' - y)z \text{ for all } y' \in R_+^{l+1} \right\}.$$

The following **facts**, regarding F, will be used in proving Lemmas 1 and 2:

1. If F is convex, then it is differentiable at y if, and only if, $\partial F(y)$ consists of a single vector; i.e. $\partial F(y) = \{\nabla F(y)\}$, where $\nabla F(y)$ is called the *gradient* of F at y.

2. If F is convex and finite in a neighborhood of y, then $\partial F(y)$ is the convex hull of the compact set

$$\{y \in R^m \mid \exists\, y_k \longrightarrow y \text{ with } F \text{ differentiable at } y_k \text{ and } \nabla F(y_k) \longrightarrow z\}.$$

3. (Euler's formula) If F is homogeneous of degree one and differentiable at y_k, then $F(y_k) = y_k \nabla F(y_k)$. Therefore, if $y_k \longrightarrow y$, with F differentiable at y_k, and $\nabla F(y_k) \longrightarrow z$, then $F(y) = yz$. In particular, $F(y)$ has (partial) directional derivatives given by: $\frac{\partial F(\mu)}{\partial^d y^j} \equiv f_j^d(y)$ – which are *extreme points* of $\partial F(\mu)$ – and $F(\mu) = \sum_{j=1}^m y^j f_j^d(y)$.

[12] As said, Marimon, Messner and Pavoni (2009) analyzes the general case with policy correspondences. We are specially grateful to the latter two authors who provided an example showing the problems that may arise when solutions are not uniquely determined. The proof of Theorem 4 has benefited from the study of their example.

4. If W is convex and homogeneous of degree one, for any pair $\left(f, \widehat{f}\right)$, if $f^d(y) \in \partial F(y)$ and $f^d(\widehat{y}) \in \partial F(\widehat{y})$, then $\widehat{y} f^d(\widehat{y}) \geq \widehat{y} f^d(y)$.

5. If F is convex, $y^* \geq 0$ satisfies $F(y) \geq F(y^*)$, for all $y \geq 0$, if and only if, for all $f(y^*) \in \partial F(y^*)$, $f(y^*) \geq 0$ and $f_j(y^*) = 0$ whenever $y^{*j} > 0$.

Fact 1 is a basic result on differentiability of convex functions (see, Rockafellar, 1981, Theorem 4F, or 1970, Theorem 25.1). Fact 2 is a very convenient characterization of the subgradient set of a convex function (see Rockafellar, 1981, Theorem 4D, or 1970, Theorem 25.6). Fact 3 is the well known Euler's formula and its second part follows from continuity and the characterization of the extreme points of the subgradient set. To see Fact 4 notice that if $F(y) = y f^d(y)$, by convexity, homogeneity of degree one, and Euler's formula,

$$\begin{aligned} F(\widehat{y}) &\geq F(y) + (\widehat{y} - y) f^d(y) \\ &= y f^d(y) + \widehat{y} f^d(y) - y f^d(y) = \widehat{y} f^d(y). \end{aligned}$$

Finally, Fact 5 is a result on minimization of convex functions on R^m_+. To see this, notice that by assumption for all $y \geq 0, F(y + y^*) - F(y^*) \geq 0$, suppose $f_j(y^*) < 0$, and let $e_j \in R^m$ be given by $e_j^k = 1$, if $k = j, e_j^k = 0$, if $k \neq j$. Then, by convexity and the fact that and $f(y^*) \in \partial F(y^*)$

$$F(e_j + y^*) - F(y^*) \geq e_j f(y^*) = f_j(y^*),$$

but if $f_j(y^*) < 0$ then

$$F(y^*) - F(e_j + y^*) \geq -f_j(y^*) > 0,$$

which contradicts the minimality of $F(y^*)$. This latter contradiction also arises if $f_j(y^*) > 0$ and $y^{*j} > 0$. To see this, let $\lambda > 0$ be small enough, such that $y^{*j} - \lambda \geq 0$, then

$$\begin{aligned} F(-\lambda e_j + y^*) - F(y^*) &\geq -\lambda e_j f(y^*) = -\lambda f_j(y^*), \\ \text{i.e. } F(y^*) - F(-\lambda e_j + y^*) &\geq f_j(y^*) > 0. \end{aligned}$$

To see the sufficiency part, notice that for all $y \geq 0$,

$$\begin{aligned} F(y) - F(y^*) &\geq (y - y^*) f(y^*) \\ &= y f(y^*) \geq 0, \end{aligned}$$

where the equality follows from the fact that $y^* f(y^*) = 0$, and the inequality from the fact that $y \geq 0$ and $f(y^*) \geq 0$.

It follows from Facts 2 and 3 that, without loss of generality, we can express the *saddle-point Bellman equation* (10) in the form

$$\begin{aligned} \mu \omega^d(x, \mu, s) &= \mu h_0(x, a^*, s) + \gamma^* h_1(x, a^*, s) \\ &\quad + \beta \text{ E}\left[\varphi(\mu, \gamma^*, s) \omega^{d'}(x^{*\prime}, \varphi(\mu, \gamma^*, s), s') \middle| s\right], \end{aligned} \tag{13}$$

where $\mu\omega^d(x,\mu,s) = W(x,\mu,s)$, and the vectors ω^d and $\omega^{d'}$ are (partial) directional derivatives of $W(x,\mu,s)$ and $W(x^{*\prime},\, \varphi(\mu,\, \gamma^*, s),\, s')$, respectively. Correspondingly, the **SPFE** *saddle-point inequalities* take the form

$$\mu h_0(x,\, a^*,\, s) + \widetilde{\gamma} h_1(x,\, a^*,\, s) + \beta\ \mathrm{E}\left[\varphi(\mu,\, \widetilde{\gamma}, s)\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \widetilde{\gamma}, s),\, s')|\, s\right]$$

$$\geq \mu h_0(x,\, a^*,\, s) + \gamma^* h_1(x,\, a^*,\, s) + \beta\ \mathrm{E}\left[\varphi(\mu,\, \gamma^*, s)\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \gamma^*, s),\, s')|\, s\right] \tag{14}$$

$$\geq \mu h_0(x,\, \widetilde{a},\, s) + \gamma^* h_1(x,\, \widetilde{a},\, s) + \beta\ \mathrm{E}\left[\varphi(\mu,\, \gamma^*, s)\omega^{d'}(\widetilde{x}',\, \varphi(\mu,\, \gamma^*, s),\, s')|\, s\right], \tag{15}$$

for any $\widetilde{\gamma} \in R_+^{l+1}$ and $(\widetilde{a}, \widetilde{x}')$ satisfying the technological constraints at $(x,\ s)$.

The *saddle-point inequality property,* **SPI**, substitutes (14) for

$$\mu h_0(x,\, a^*,\, s) + \widetilde{\gamma} h_1(x,\, a^*,\, s) + \beta\ \mathrm{E}\left[\varphi(\mu,\, \widetilde{\gamma}, s)\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \gamma^*, s),\, s')|\, s\right]$$

$$\geq \mu h_0(x,\, a^*,\, s) + \gamma^* h_1(x,\, a^*,\, s) + \beta\ \mathrm{E}\left[\varphi(\mu,\, \gamma^*, s)\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \gamma^*, s),\, s')|\, s\right] \tag{16}$$

We first show, in Lemma 1, the equivalence between **SPFE** and **SPI.**

Lemma 1 (SPI $\Longleftrightarrow$ SPFE). If $W(x,\cdot,s)$ is convex and homogeneous of degree one, then (14) is satisfied if and only if (16) is satisfied. Furthermore, the inequality (16) is characterized by the following first order necessary and sufficient conditions, for $j = 0, ..., l$,

$$h_1^j(x,\, a_0^*,\, s) + \beta\ \mathrm{E}\left[\omega_j^{d'}(x_1^*,\, \mu_1^*,\, s_1)|\, s\right] \quad \geq \quad 0 \tag{17}$$

$$\gamma^{*j}\left[h_1^j(x,\, a_0^*,\, s) + \beta\ \mathrm{E}\left[\omega_j^{d'}(x_1^*,\, \mu_1^*,\, s_1)|\, s\right]\right] \quad = \quad 0. \tag{18}$$

Proof of Lemma 1: That **SPI $\Longrightarrow$ SPFE** follows from Fact 4. With respect to $W(x^{*\prime},\, \varphi(\mu,\, \gamma, s),\, s')$, Fact 4 takes the form:

$$\varphi(\mu,\, \widetilde{\gamma}, s)\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \widetilde{\gamma}, s),\, s') \geq \varphi(\mu,\, \widetilde{\gamma}, s)\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \gamma^*, s),\, s');$$

therefore (16) together with this latter inequality results in the following inequalities, which show that (14) is satisfied whenever (16) is satisfied:

$$\begin{aligned} & \mu h_0(x,\, a^*,\, s) + \widetilde{\gamma} h_1(x,\, a^*,\, s) + \beta\ \mathrm{E}\left[\varphi(\mu,\, \widetilde{\gamma}, s)\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \widetilde{\gamma}, s),\, s')|\, s\right] \\ \geq\ & \mu h_0(x,\, a^*,\, s) + \widetilde{\gamma} h_1(x,\, a^*,\, s) + \beta\ \mathrm{E}\left[\varphi(\mu,\, \widetilde{\gamma}, s)\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \gamma^*, s),\, s')|\, s\right] \\ \geq\ & \mu h_0(x,\, a^*,\, s) + \gamma^* h_1(x,\, a^*,\, s) + \beta\ \mathrm{E}\left[\varphi(\mu,\, \gamma^*, s)\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \gamma^*, s),\, s')|\, s\right] \end{aligned}$$

To see that **SPFE $\Longrightarrow$ SPI,** let

$$G_{(x,\, a^*,\, s)}(\gamma,\mu) \equiv \mu h_0(x,\, a^*,\, s) + \gamma h_1(x,\, a^*,\, s) + \beta\, \mathrm{E}\left[\varphi(\mu,\, \gamma, s)\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \gamma, s),\, s')|\, s\right],$$

and

$$F_{(x,\, a^*,\, s)}(\gamma, \mu) \equiv \mu h_0(x,\, a^*,\, s) + \gamma h_1(x,\, a^*,\, s) + \beta \,\mathrm{E}\left[\varphi(\mu,\, \gamma, s)\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \gamma^*, s),\, s') |\, s\right],$$

Then (14) reduces to $G_{(x,\, a^*,\, s)}(\gamma, \mu) \geq G_{(x,\, a^*,\, s)}(\gamma^*, \mu)$, (16) to $F_{(x,\, a^*,\, s)}(\gamma, \mu) \geq F_{(x,\, a^*,\, s)}(\gamma^*, \mu)$ and, since $G_{(x,\, a^*,\, s)}(\gamma^*, \mu) = F_{(x,\, a^*,\, s)}(\gamma^*, \mu)$, the above inequalities show that, if $f_{(x,\, a^*,\, s)}(\gamma^*, \mu) \in \partial_\gamma F_{(x,\, a^*,\, s)}(\gamma^*, \mu)$, for all $\gamma \geq 0$,

$$\begin{aligned} G_{(x,\, a^*,\, s)}(\gamma, \mu) - G_{(x,\, a^*,\, s)}(\gamma^*, \mu) \quad &\geq \quad F_{(x,\, a^*,\, s)}(\gamma, \mu) - F_{(x,\, a^*,\, s)}(\gamma^*, \mu) \\ &\quad\quad (\gamma - \gamma^*)\, f_{(x,\, a^*,\, s)}(\gamma^*, \mu); \end{aligned}$$

that is, $f_{(x,\, a^*,\, s)}(\gamma^*, \mu) \in \partial_\gamma G_{(x,\, a^*,\, s)}(\gamma^*, \mu)$. However, if $g_{(x,\, a^*,\, s)}(\gamma^*, \mu)$ is an extreme point of $\partial_\gamma G_{(x,\, a^*,\, s)}(\gamma^*, \mu)$, then

$$g_{(x,\, a^*,\, s)}(\gamma^*, \mu) = \left[h_1(x,\, a^*(x,\, \mu,\, s),\, s) + \beta \,\mathrm{E}\left[\omega^{d'}(x^{*\prime}(x,\, \mu,\, s),\, \varphi(\,\mu, \gamma^*,\, s),\, s') |\, s\right]\right],$$

and, therefore, $g_{(x,\, a^*,\, s)}(\gamma^*, \mu) \in \partial_\gamma F_{(x,\, a^*,\, s)}(\gamma^*, \mu)$ – in fact, it is also an extreme point of $\partial_\gamma F_{(x,\, a^*,\, s)}(\gamma^*, \mu)$. This shows that $\partial_\gamma F_{(x,\, a^*,\, s)}(\gamma^*, \mu) = \partial_\gamma G_{(x,\, a^*,\, s)}(\gamma^*, \mu)$, which, in turn, implies the equivalence between (14) and (16).

Finally, (17) and (18) are an immediate consequence of Fact 5■

Lemma 2. If the assumptions of Theorem 4 are satisfied, then $W(x, \cdot, s)$ is differentiable at μ.

Proof of Lemma 2: By Facts 2 and 3, we can consider that W takes the form $W(x,\, \mu,\, s) = \mu\omega(x,\, \mu,\, s)$, where $\omega(x,\, \mu,\, s)$ is uniquely defined (i.e. $\omega(x,\, \mu,\, s) \equiv \nabla_\mu W(x,\, \mu,\, s)$) if, and only if, $W(x, \cdot, s)$ is differentiable at μ. It is convenient to consider first the case of one-period constraints (i.e. $j = k+1, ..., l$). In this case the saddle-point Bellman equation, corresponding to **SPFE,** takes the form:

$$\mu\omega^d(x,\, \mu,\, s) = \mu h_0(x, a^*,\, s) + \gamma^* \left[h_1(x, a^*, s) + \beta \mathrm{E}\left[\omega^{d'}(x^{*\prime}, \gamma^*,\, s') \mid s\right]\right]. \tag{19}$$

Define $F_{(x,\, a^*,\, s)}(\gamma, \mu)$ as in the proof of Lemma 1; that is, $F_{(x,\, a^*,\, s)}(\gamma^*, \mu)$ is the right-hand side of (19). Then,

$$\begin{aligned} \frac{\partial F_{(x,\, a^*,\, s)}(\gamma^*, \mu)}{\partial \mu^j} \quad &= \quad h_0^j(x, a^*,\, s) + \frac{\partial F_{(x,\, a^*,\, s)}(\gamma^*, \mu)}{\partial \gamma^*}\frac{\partial \gamma^*}{\partial \mu^j} + \frac{\partial F_{(x,\, a^*,\, s)}(\gamma^*, \mu)}{\partial a^*}\frac{\partial a^*}{\partial \mu^j} \\ &= \quad h_0^j(x, a^*,\, s) \end{aligned}$$

That $\frac{\partial F_{(x,\, a^*,\, s)}(\gamma^*, \mu)}{\partial \gamma^*} = 0$ follows from the fact that $F_{(x,\, a^*,\, s)}(\gamma, \mu)$ is differentiable in γ, together with (17) and (18); that $\frac{\partial F_{(x,\, a^*,\, s)}(\gamma^*, \mu)}{\partial a^*} = 0$ follows from the fact that, under the assumptions of Theorem 4, $W(\cdot,\, \mu,\, s)$

satisfies the conditions of Benveniste and Scheinkman' theorem on the differentiability of concave value functions (Stokey, Lucas and Prescott, 1989, Theorem 4.10) and, therefore, (15) is also characterized by first-order conditions. In summary, since $\frac{\partial W(\mu)}{\partial^d \mu_j} \equiv \omega_j^d(\mu)$, this *envelope theorem* argument shows that, for $j = k+1, ..., l$, $\omega_j^d(x,\, \mu,\, s) = h_0^j(x, a^*,\, s)$. The latter equalities imply that $\omega_j^d(x,\, \mu,\, s)$ is uniquely determined whenever a^* is the unique maximizer. It then follows from Fact 1 that $W(x, \cdot, s)$ is differentiable at μ.

Consider next the case of intertemporal participation constraints (i.e. $j = 0, ..., k$). In this case the saddle-point Bellman equation, corresponding to **SPFE,** takes the form:

$$\begin{aligned} \mu\omega^d(x,\, \mu,\, s) &= \mu \left[h_0(x, a^*,\, s) + \beta \mathrm{E}\left[\omega^{d'}(x^{*\prime}, \mu + \gamma^*,\, s') \mid s\right]\right] \\ &\quad +\gamma^* \left[h_1(x, a^*,\, s) + \beta \mathrm{E}\left[\omega^{d'}(x^{*\prime}, \mu + \gamma^*,\, s') \mid s\right]\right] \end{aligned}$$

Using, as before, an envelope argument and the fact that the (partial) directional derivatives are well – but, possibly, not uniquely – defined, we obtain

$$\omega_j^d(x,\, \mu,\, s) = h_0^j(x, a^*,\, s) + \beta \mathrm{E}\left[\omega_j^{d'}(x^{*\prime}, \mu + \gamma^*,\, s') \mid s\right] \tag{20}$$

In particular, non differentiability of W at μ means that – at least from some j – the right and the left (partial) derivatives[13] do not coincide: $\omega_j^+(x,\, \mu,\, s) \neq \omega_j^-(x,\, \mu,\, s)$. By recursive iteration of (20), and using the fact that W is bounded, we obtain

$$\omega_j^d(x,\, \mu,\, s) = \mathrm{E}_0\left[\sum_{t=0}^{\infty} \beta^t h_0^j(x_t^*,\, a_t^*,\, s_t) | \, s\right], \text{ if } j = 0, ..., k. \tag{21}$$

Uniqueness of the solution $\{x_t^*,\, a_t^*,\, s_t\}_{t=0}^{\infty}$ implies that the left-hand side of (21) is uniquely determined; that is, $\omega_j(x,\, \mu,\, s) \equiv \omega_j^+(x,\, \mu,\, s) = \omega_j^-(x,\, \mu,\, s)$. From Facts 2 and 1, the differentiability of $W(x, \cdot, s)$ at μ follows■

Notice that by **(18)** the *aggregate saddle-point Bellman equation* (13) simplifies to

$$\mu\omega(x, \mu, s) = \mu \left[h_0(x,\, a^*(x,\, \mu,\, s),\, s) + \beta \ \mathrm{E}\left[\chi^j \omega(x^{*\prime}(x,\, \mu,\, s),\, \varphi(x,\, \mu,\, s),\, s') | \, s\right]\right],$$

where $\chi^j = 1$, if $j = 0, ..., k, \chi^j = 0$, if $j = k+1, ..., l$. The proof of Lemma 2 shows that, under the assumptions of Theorem 4, the *aggregate saddle-point Bellman equation* also translates into (componentwise) *individual saddle-point Bellman equations*. Formally,

[13] The left (partial) derivative is defined as $\omega_j^-(x,\, \mu,\, s) \equiv \frac{\partial W(x,\mu,s)}{\partial^- \mu_i} = \lim_{\lambda \downarrow 0} \frac{W(x, \mu - \lambda e_i, s) - W(x,\mu,s)}{\lambda}$.

Corollary: The following (recursive) equations are satisfied:

$$\omega_j(x,\,\mu,\,s) = h_0^j(x,\,a^*(x,\,\mu,\,s),\,s){+}\beta\,\mathrm{E}\left[\omega_j(x^{*\prime}(x,\,\mu,\,s),\,\mu^{*\prime}(x,\,\mu,\,s),\,s')|\,s\right], \tag{22}$$

if $j = 0, ..., k$,

$$\omega_j(x,\,\mu,\,s) = h_0^j(x,\,a^*(x,\,\mu,\,s),\,s),\ \text{if}\ j = k+1, ..., l. \tag{23}$$

We now turn to the proof of Theorem 4.

Proof (Theorem 4): By Lemma 2, there is a unique representation $W(x,\,\mu,\,s) = \mu\omega(x,\,\mu,\,s)$, where $\omega_j(x,\,\mu,\,s) \equiv \frac{\partial W(x,\mu,s)}{\partial \mu_i}$. To see that solutions of **SPFE** satisfy the participation constraints of $\mathbf{SPP}_\mu$, we use the first order conditions (17) and 18), as well as the individual recursive equations (22) and (23). As in the proof of Lemma 2, equation (22) can be iterated to obtain

$$\omega_j(x,\,\mu,\,s) = \mathrm{E}_0\left[\sum_{t=0}^{\infty}\beta^t h_0^j(x_t^*,\,a_t^*,\,s_t)|\,s\right],\ \text{if}\ j = 0, ..., k. \tag{24}$$

Following the same steps for any $t > 0$ and state $(x_t^*,\,\mu_t^*,\,s_t)$, equation(23) and (24) together with the inequality (17) show that the intertemporal participation constraints in $\mathbf{PP}_\mu$ – and therefore in $\mathbf{SPP}_\mu$ – are satisfied; that is,

$$\mathrm{E}_t\sum_{n=1}^{N_j+1}\beta^n\ h_0^j(x_{t+n}^*,\,a_{t+n}^*,\,s_{t+n}) + h_1^j(x_t^*,\,a_t^*,\,s_t) \geq 0,\ ;\ t \geq 0, \quad j = 0, ..., l \tag{25}$$

Notice that equations (23) and (24) also show that

$$\mu\omega(x,\,\mu,\,s) = \mathrm{E}_0\sum_{j=0}^{l}\sum_{t=0}^{N_j}\beta^t\mu^j h_0^j(x_t^*,\,a_t^*,\,s_t) \tag{26}$$

Finally, suppose there exist a program $\{\widetilde{a}_t\}_{t=0}^{\infty}$, and $\{\widetilde{x}_t\}_{t=0}^{\infty}$, $\widetilde{x}_0 = x, \widetilde{x}_{t+1} = \ell(\widetilde{x}_t,\,\widetilde{a}_t,\,s_{t+1})$, satisfying the constraints of $\mathbf{SPP}_\mu$ with initial condition (x, s) and such that

$$\begin{aligned}
&\mu h_0(x,\,\widetilde{a}_0,\,s) + \gamma^* h_1(x,\,\widetilde{a}_0,\,s)\\
&+\beta\mathrm{E}\left[\sum_{j=0}^{k}\left(\mu^j+\gamma^{*j}\right)\sum_{n=1}^{\infty}\beta^t\ h_0^j(\widetilde{x}_t,\,\widetilde{a}_t,\,s_t) + \sum_{j=k+1}^{l}\gamma^{*j}h_0^j(\widetilde{x}_1,\,\widetilde{a}_1,\,s_1)|\,s\right]\\
&>\ \mu h_0(x,\,a_0^*,\,s) + \gamma^* h_1(x,\,a_0^*,\,s)\\
&+\beta\mathrm{E}\left[\sum_{j=0}^{k}\left(\mu^j+\gamma^{*j}\right)\sum_{n=1}^{\infty}\beta^t\ h_0^j(x_t^*,\,a_t^*,\,s_t) + \sum_{j=k+1}^{l}\gamma^{*j}h_0^j(x_1^*,\,a_1^*,\,s_1)|\,s\right].
\end{aligned} \tag{27}$$

The following string of equalities and inequalities, which we explain at the end, contradict this inequality,

$$\mu h_0(x,\, a_0^*,\, s) + \gamma_0^* h_1(x,\, a_0^*,\, s)$$
$$+ \beta \mathrm{E}\left[\sum_{j=0}^{k} \left(\mu^j + \gamma^{*j}\right) \sum_{n=1}^{\infty} \beta^t \; h_0^j(x_t^*,\, a_t^*,\, s_t) + \sum_{j=k+1}^{l} \gamma^{*j} h_0^j(x_1^*,\, a_1^*,\, s_1) |\, s \right]$$
$$= \mu h_0(x,\, a_0^*,\, s) + \gamma_0^* h_1(x,\, a_0^*,\, s) + \beta\ \mathrm{E}\left[\mu_1^* \omega(x_1^*,\, \mu_1^*,\, s_1) |\, s\right] \tag{28}$$
$$\geq \mu h_0(x,\, \widetilde{a}_0,\, s) + \gamma_0^* h_1(x,\, \widetilde{a}_0,\, s) + \beta\ \mathrm{E}\left[\mu_1^* \omega(\widetilde{x}_1,\, \mu_1^*,\, s_1) |\, s\right] \tag{29}$$
$$= \mu h_0(x,\, \widetilde{a}_0,\, s) + \gamma_0^* h_1(x,\, \widetilde{a}_0,\, s)$$
$$+ \beta\ \mathrm{E}\left[\mu_1^* h_0(\widetilde{x}_1,\, a^*(\widetilde{x}_1,\, \mu_1^*, s_1),\, s_1) + \gamma^*(\widetilde{x}_1,\, \mu_1^*, s_1) h_1(\widetilde{x}_1,\, a^*(\widetilde{x}_1,\, \mu_1^*, s_1),\, s_1)\right. \tag{30}$$
$$\left. +\ \beta \mu^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1) \omega(x^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1),\, \mu^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1), s_2) |\, s\right]$$
$$\geq \mu h_0(x,\, \widetilde{a}_0,\, s) + \gamma_0^* h_1(x,\, \widetilde{a}_0,\, s)$$
$$+ \beta\ \mathrm{E}\left[\mu_1^* h_0(\widetilde{x}_1,\, \widetilde{a}_1,\, s_1) + \gamma^*(\widetilde{x}_1,\, \mu_1^*, s_1) h_1(\widetilde{x}_1,\, \widetilde{a}_1,\, s_1)\right. \tag{31}$$
$$\left. +\ \beta \mu^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1) \omega(\widetilde{x}_2,\, \mu^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1), s_2) |\, s\right]$$
$$\geq \mu h_0(x,\, \widetilde{a}_0,\, s) + \gamma_0^* h_1(x,\, \widetilde{a}_0,\, s)$$
$$+ \beta\ \mathrm{E}\left[\mu_1^* h_0(\widetilde{x}_1,\, \widetilde{a}_1,\, s_1) + \gamma^*(\widetilde{x}_1,\, \mu_1^*, s_1) h_1(\widetilde{x}_1,\, \widetilde{a}_1,\, s_1) + \beta \mu^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1) \omega(\widetilde{x}_2,\ \mu_1^*, s_2) |\, s\right] \tag{32}$$
$$\geq \mu h_0(x,\, \widetilde{a}_0,\, s) + \gamma_0^* h_1(x,\, \widetilde{a}_0,\, s)$$
$$+ \beta\ \mathrm{E}\left[\mu_1^* \left[h_0(\widetilde{x}_1,\, \widetilde{a}_1,\, s_1) + \beta \omega(\widetilde{x}_2,\ \mu_1^*, s_2)\right] |\, s\right] \tag{33}$$
$$= \mu h_0(x,\, \widetilde{a}_0,\, s) + \gamma_0^* h_1(x,\, \widetilde{a}_0,\, s) + \beta\ \mathrm{E}\left[\mu_1^* h_0(\widetilde{x}_1,\, \widetilde{a}_1,\, s_1) |\, s\right]$$
$$+ \beta^2\ \mathrm{E}\left[\mu_1^* h_0(\widetilde{x}_2,\, a^*(\widetilde{x}_2,\, \mu_1^*, s_2),\, s_2) + \gamma^*(\widetilde{x}_2,\, \mu_1^*, s_2) h_1(\widetilde{x}_2,\, a^*(\widetilde{x}_2,\, \mu_1^*, s_2),\, s_2)\right. \tag{34}$$
$$\left. +\ \beta \mu^{*\prime}(\widetilde{x}_2,\, \mu_1^*, s_2) \omega(x^{*\prime}(\widetilde{x}_2,\, \mu_1^*, s_2),\, \mu^{*\prime}(\widetilde{x}_2,\, \mu_1^*, s_2), s_2) |\, s\right]$$
$$\geq \mu h_0(x,\, \widetilde{a}_0,\, s) + \gamma_0^* h_1(x,\, \widetilde{a}_0,\, s)$$
$$+ \beta \mathrm{E}\left[\sum_{j=0}^{k} \left(\mu^j + \gamma^{*j}\right) \sum_{n=1}^{\infty} \beta^t \; h_0^j(x_t^*,\, a_t^*,\, s_t) + \sum_{j=k+1}^{l} \gamma^{*j} h_0^j(x_1^*,\, a_1^*,\, s_1) |\, s \right] \tag{35}$$

Notice that the first equality (28) is just notational and the other two equalities (30) and (34) are simple expansions of the of the saddle-point value paths (i.e., of (13)). Inequalities (29) and (31) follow from the maximality property of **SPFE.** Inequalities (32) and (33) require explanation. Inequality (32) follows from using the above stated fact about convex and homogeneous of degree one functions (i.e. Fact 4: $\widehat{\mu}\omega(\widehat{\mu}) \geq \widehat{\mu}\omega(\mu)$) since (32) is simply $\mu^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1)\omega(\widetilde{x}_2,\, \mu^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1) \geq \mu^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1)\omega(\widetilde{x}_2,\ \mu_1^*, s_2)$. Inequality (33) follows from applying the slackness inequality (17), as well as equations (23) and (24) to the plan generated by **SPFE** in state $(\widetilde{x}_2,\ \mu_1^*, s_2)$ (i.e. to $\{a_t^*(\widetilde{x}_2,\ \mu_1^*, s_2)\}_{t=2}^{\infty}$), showing that such plan satisfies the corresponding **SPP** constraints (25); that is, $\left[h_1^j(\widetilde{x}_1,\, \widetilde{a}_1,\, s_1) + \beta\omega_j(\widetilde{x}_2,\ \mu_1^*, s_2)\right] \geq 0,\ j = 0, ..., l.$

Finally, since the equality (34) is simply the equality (30) after one iteration, repeated iterations result in the last inequality (35), which contradicts (27).

It only remains to show that the inf part of **SPP** is also satisfied. Reasoning again by contradiction, suppose there exist a $\widetilde{\gamma} \geq 0$ such that

$$\begin{aligned} & \mu h_0(x,\, a_0^*,\, s) + \widetilde{\gamma} h_1(x,\, a_0^*,\, s) \\ & + \beta \mathrm{E}\left[\sum_{j=0}^{k} \left(\mu^j + \widetilde{\gamma}^j \right) \sum_{n=1}^{\infty} \beta^t \; h_0^j(x_t^*,\, a_t^*,\, s_t) + \sum_{j=k+1}^{l} \widetilde{\gamma}^j h_0^j(x_1^*,\, a_1^*,\, s_1) |\, s \right] \\ < \; & \mu h_0(x,\, a_0^*,\, s) + \gamma^* h_1(x,\, a_0^*,\, s) \\ & + \beta \mathrm{E}\left[\sum_{j=0}^{k} \left(\mu^j + \gamma^{*j} \right) \sum_{n=1}^{\infty} \beta^t \; h_0^j(x_t^*,\, a_t^*,\, s_t) + \sum_{j=k+1}^{l} \gamma^{*j} h_0^j(x_1^*,\, a_1^*,\, s_1) |\, s \right]. \end{aligned} \tag{36}$$

Using (26), this inequality can also be expressed as

$$\begin{aligned} & \widetilde{\gamma} \left[h_1(x,\, a^*(x,\, \mu,\, s),\, s) + \beta \; \mathrm{E}\left[\omega(x^{*\prime}(x,\, \mu,\, s),\, \mu^{*\prime}(x,\, \mu,\, s),\, s') |\, s \right] \right] \\ < \quad & \gamma^*(x,\, \mu,\, s) \left[h_1(x,\, a^*(x,\, \mu,\, s),\, s) + \beta \; \mathrm{E}\left[\omega_j(x^{*\prime}(x,\, \mu,\, s),\, \mu^{*\prime}(x,\, \mu,\, s),\, s') |\, s \right] \right], \end{aligned}$$

but the first order conditions (17) and (18) require that (16) is satisfied, i.e.

$$\begin{aligned} & \widetilde{\gamma} \left[h_1(x,\, a^*(x,\, \mu,\, s),\, s) + \beta \; \mathrm{E}\left[\omega(x^{*\prime}(x,\, \mu,\, s),\, \mu^{*\prime}(x,\, \mu,\, s),\, s') |\, s \right] \right] \\ \geq \quad & \gamma^*(x,\, \mu,\, s) \left[h_1(x,\, a^*(x,\, \mu,\, s),\, s) + \beta \; \mathrm{E}\left[\omega_j(x^{*\prime}(x,\, \mu,\, s),\, \mu^{*\prime}(x,\, \mu,\, s),\, s') |\, s \right] \right] = 0 \end{aligned}$$

which contradicts (36)■

A key step in the proof of Theorem 4 is to show that the representation of W, given by Euler's Theorem (Fact 3), is unique and satisfies the (componentwise) *individual saddle-point Bellman equations*, which are needed to show that the constraints of the $\mathbf{PP}_\mu$ are satisfied. The proof relies on Benveniste and Scheinkman's theorem on the differentiability of concave value functions, which is used to establish an *envelope theorem* argument. However, one can establish the latter argument with alternative assumptions (e.g. guaranteeing differentiability of W in x), or once $W = \mu\omega$ is known it may be relatively easy to establish the uniqueness of the representation of W, as well as that *individual saddle-point Bellman equations* are satisfied. The following Corollary makes this remark more precise.

Corollary (Sufficiency): Assume W, satisfying **SPFE,** is continuous in (x, μ) and convex and homogeneous of degree one in μ**.** Then if $(\mathbf{a}^*, \boldsymbol{\gamma}^*)$ is generated by the *SP policy function* ψ, associated with W, from an initial condition (x, μ, s) and, for all $(t, s_t), t \geq 0$, there is a unique representation $W(x_t^*, \mu_t^*, s_t) = \mu_t^* \omega(x_t^*, \mu_t^*, s_t)$, satisfying (22) and (23), then $(\mathbf{a}^*, \boldsymbol{\gamma}^*)$ is also a solution to $\mathbf{SPP}_\mu$ at (x, s).

5 DSPP and the contraction mapping

In this Section we sharpen the results of Theorem 4 by applying the *Contraction Mapping Theorem* to the *Dynamic Saddle-Point Problem.* We fully exploit the properties of the value functions derived in the previous section. We first define the space of "value" functions,

$$M = \{\omega : X \times \mathcal{R}_+^{l+1} \times S \to \mathcal{R}^{l+1} \text{ s.t., for } j = 0, ..., l,$$

$$\begin{aligned} &i)\ \omega_j(\cdot,\,\cdot,\,s) \text{ is continuous, and } \omega_j(\cdot,\,\mu,\,s) \text{ is bounded if } \|\mu\| \leq 1 \\ &ii)\ \omega_j(\cdot,\,\mu,\,s) \text{ is concave, and} \\ &iii)\ \omega_j(x,\,\cdot,\,s) \text{ is convex and homogeneous of degree zero}\} \end{aligned}$$

The space M is a normed vector space with the norm

$$\|\omega\| = \sup \left\{ |\omega_j(x, \mu, s)| : j = 0, ..., l, \|\mu\| \leq 1, x \in X, s \in S \right\},$$

and we show in the Appendix (Lemma 3A) that it is a complete metric space; therefore, a suitable space for the *Contraction Mapping Theorem.* Given a value function $W = \mu\omega$, $\omega \in M$, we can define the following *Dynamic Saddle Point Problem*:

$\mathbf{DSPP}_{(x,\mu,s)}$

$$\inf_{\gamma \geq 0} \sup_{a} \left\{ \mu h_0(x,\,a,\,s) + \gamma h_1(x,\,a,\,s) + \beta\ \mathrm{E}\left[\mu'\omega(x',\,\mu',\,s')|\,s\right] \right\}$$

$$\text{s.t. } x' = \ell(x,\,a,\,s),\ p(x, a,\,s) \geq 0$$

$$\text{and } \mu' = \varphi(\mu,\,\gamma,\,s),$$

In order for this problem to have well defined solutions we must make an interiority assumption:

A7b. For any $(x, \mu, s) \in X \times R_+^{l+1} \times S$, there exists an $\widetilde{a} \in A$, satisfying **A2**, such that, for any $\mu' \in R_+^{l+1}, \|\mu'\| \leq 1$, and $j = 0, ..., l$: $h_1^j(x, \widetilde{a}, s) + \beta \mathrm{E}\left[\omega^j(\ell(x,\,\widetilde{a},\,s'), \mu', s')|\,s\right] > 0$.

Notice that **A7b** is satisfied, whenever **A7** is satisfied and $\mu'\omega(\ell(x,\,\widetilde{a},\,s'), \mu', s')$ is the value function of $\mathbf{SPP}_{(\ell(x,\,\widetilde{a},\,s'),\mu',s')}$. In general, **A7b** is not a restrictive assumption in the class of possible value functions if the original problem has interior solutions. Nevertheless, an assumption, such as **A7**b is needed when one takes $\mathbf{DSPP}_{(x,\mu,s)}$ as the starting problem. This is a relatively standard min max problem, except for the dependency of ω on $\varphi(\mu,\,\gamma,\,s)$. The following proposition shows that it has a solution. Obviously, solutions to $\mathbf{DSPP}_{(x,\mu,s)}$ satisfy **SPFE.**

Proposition 2. Let $\omega \in M$ and assume **A1-A6** and **A7b.** There exists (a^*, γ^*) that solves $\mathbf{DSPP}_{(x,\mu,s)}$. Furthermore if **A6s** is assumed, then $a^*(x, \mu, s)$ is uniquely determined.

Proof: See Appendix.

A8. The action $a^*(x, \mu, s)$, solving $\mathbf{DSPP}_{(x,\mu,s)}$, is uniquely determined.

$\mathbf{DSPP}_{(x,\mu,s)}$ defines then a **SPFE** operator, $T : M \longrightarrow M$, given by

$$\omega_j(x,\, \mu,\, s) = h_0^j(x,\, a^*(x,\, \mu,\, s),\, s) + \beta\, \mathrm{E}\left[\omega_j(x^{*\prime}(x,\, \mu,\, s),\, \mu^{*\prime}(x,\, \mu,\, s),\, s')\right| s] \,, \tag{37}$$

if $j = 0, ..., k$, and

$$\omega_j(x,\, \mu,\, s) = h_0^j(x,\, a^*(x,\, \mu,\, s),\, s), \text{ if } j = k+1, ..., l.$$

Let M^k be the $j = 0, ..., k$, projection of M and $T_k : M^k \to M^k$ be the operator defined by (37), then under our assumptions 'all individual values ω_j are uniquely determined'. More formally,

Lemma 3. Assume **A1-A6, A6b, A7b** and **A8.** $T_k : M^k \to M^k$ is a well defined contraction mapping.

Proof: The proof is an immediate consequence of Lemmas 3A to 7A in the Appendix.

As we have seen, since 'individual values ω_i are the are the subgradients W', Lemma 3 also shows the differentiability of W in μ. More importantly, another immediate consequence of Lemma 3, and our previous Theorems 4 and 2, is our main final theorem:

Theorem 5 ($\mathbf{DSPP}_{(x,\mu,s)} \Longrightarrow \mathbf{PP}_\mu(x,s)$). Assume **A1-A6, A6b, A7b** and **A8**. $T : M \to M$ has a unique solution ω, which defines a *value function* $W(x, \mu, s) = \mu\omega(x, \mu, s)$ and a *saddle-point policy function* ψ, such that if $(\mathbf{a}^*, \boldsymbol{\gamma}^*)$ is generated by ψ from $(x,\, \mu,\, s)$, then $\mathbf{a}^*$ is the unique solution to $\mathbf{PP}_\mu$ at $(x,\, s)$.

As we have emphasized not all our assumptions are necessary. In particular, in applications assumptions, such as concavity of the h functions may not be satisfied and yet $\mathbf{DSPP}_{(x,\mu,s)}$ may have unique solutions. The following Corollary, makes this remark more precise. Let $\widetilde{M}$ be defined as M, without imposing *ii*) (i.e, not requiring concavity).

Corollary (Bounded returns): Assume **A1-A5** and that, for all $(x,\, \mu,\, s)$, $\mathbf{DSPP}_{(x,\mu,s)}$ has a solution, unique in $a^*(x, \mu, s)$, then $T : \widetilde{M} \to \widetilde{M}$ has a unique solution ω, which defines a *value function* $W(x, \mu, s) = \mu\omega(x, \mu, s)$ and a *saddle-point policy function* ψ, such that if $(\mathbf{a}^*, \boldsymbol{\gamma}^*)$ is generated by ψ from $(x,\, \mu,\, s)$, then $\mathbf{a}^*$ is the unique solution to $\mathbf{PP}_\mu$ at $(x,\, s)$.

6 Related work

Precedents of our approach can be found in Kydland and Prescott (1980) and in Hansen, *et al.* (1985). They introduced lagrange multipliers as co-state variables. Our work provides a formal proof that a stationary optimal policy can be obtained by properly introducing lagrange multipliers as co-states. There are special cases, however, where there is a one-to-one relationship between states and co-states. Obviously, in these cases it is possible to obtain a policy function that only depends on the state variables, although it may be discontinuous[14]. Rustichini (1998) provides a recursive characterization that encompasses these cases. He focuses on deterministic models with one (default) constraint. In this class of models, in general, it is possible to reduce the dimensionality of the policy to its natural states[15]. Cooley, Quadrini and Marimon (2000) obtain a similar result in a stochastic model with possible default and risk neutral agents. Efficient contracts in these models have the special feature that if a state is reached there can no be different past promises, regarding the value of the contract at that state, depending on past contractual histories. In deterministic models past histories may be uniquely determined by the state and in models with risk neutral agents there is no need for consumption smoothing. However, in economies with uncertainty and risk-averse agents the effect of intertemporal (default) constraints must be *smoothed* and, as a result, since the same state may be reached following different contractual histories the optimal contract can not have a unique value associated with such state[16].

The pioneer work of Abreu, *et al.* (1990) -APS, from now on,- characterizing sub-game perfect equilibria, shows that past histories can be summarized in terms of promised utilities (we summarize them with the co-state μ). While the pioneer work of Green (1987) and Thomas and Worral (1988)) -GTW, from now on- shows that efficient contracts, promising a given initial level of (present value) utility, can have a recursive structure. These related approaches have been widely used in macroeconomics[17]. In particular, Kocherlakota (1995) has applied GTW to characterize optimal social insurance contracts with participation constraints[18] and the APS approach has been further developed by Cronshaw and Luenberger (1994), to study dynamic games, and by Chang (1998)

[14] In particular, discontinuities may arise at points where, given a value of the state variable, a co-state variable jumps. In contrast, the corresponding policy function of our approach (i.e., a function of states and co-states) may well be continuous. An example of such discontinuities can be found in Benhabib and Rustichini (1996).

[15] Notice that a similar "reduction" is proposed by the Subspace Method to solve, for example, Optimal Linear Regulator problems (see, for example, Hansen and Sargent (2000)). However, as the Subspace Method shows, even when the "reduction" is possible, can be very convenient to express policies in terms of state and co-state variables.

[16] See Marcet and Marimon (1992) or Kocherlakota (1995) for an explanation of how, to achieve consumption smoothing, idiosyncratic shocks imply increased consumption over the whole future. This can only be achieved by keeping track of past shocks through μ.

[17] Ljungqvist and Sargent (2000) provides an excellent introduction, and reference, to most of this recent work.

[18] A two agents (without capital) version of Marcet and Marimon (1992), recently studied using our approach by Attanasio and Rios-Rull (2000).

and Phelan and Stacchetti (1999), to study Ramsey models.

There are similarities and differences between our approach and the "promised utilities" approaches. The extent that these approaches duplicate, complement or dominate, each other will not be fully understood until they are more developed and applied[19]. Nevertheless, some conceptual differences emerge. For instance, our approach provides a common framework that encompasses two-period (dynamic competitive) constraints as well as discounted sums that may or may not be discounted utilities (as is the case with some present value budget constraints). Within this general framework, our approach directly characterizes efficient contracts, without having to find the whole set of feasible -incentive compatible- contracts. Presumably one could develop APS or GTW to provide a similar general framework (and the work of Chang (1998) and Phelan and Stacchetti (1999) are important steps in this direction), but it may not be the most efficient way to proceed. There are issues of dimensionality and *proper recursivity* that must be taken into account.

By *proper recursivity* we mean that our SPFE provides a recursive solution that starts out from pre-set initial conditions[20] In the GTW approach initial promised (present value) utilities must be specified for all -but one- agents and, therefore, needs to be known that it is feasible. As it is well known from standard Pareto Optimal problems such approach is very useful when there are two agents and the Pareto frontier is downward slopping, otherwise it can become very cumbersome[21]. Similarly, while the APS approach provides a method to obtain the set of feasible initial present values[22] in many applications may be a fairly roundabout way to proceed. For instance, one could apply the same method to solve a simple dynamic model -such as the neoclassical growth model-where interetemporal constraints are given by Euler equations. That is, given a "promised expected marginal utility" for next period, the Euler equation determines which current actions and marginal utilities are feasible. Proper iteration of such a map determines which initial conditions result in paths satisfying the *transversality conditions* and, therefore, the initial marginal utility that an efficient planner must set (given an initial capital and shock). Standard recursive methods, aimed at obtaining directly the value and policy functions, have proved to be a more useful approach for problems of this type. We think that

[19] For instance, our approach needs to be fully developed to incorporate private information constraints and its range of applications is yet to be exploited. Similarly, some of our assumptions -such as the linear additivity of utilities and constraints- need to be relaxed in order to make its generality at par with some of the developments of Rustichini, APS or GTW approaches. We leave these issues for future research (see, however, Footnote 2).

[20] For example, when we transform the problem **PP** into the problem **SPP** we set $\overline{\mu} = (1, 0, ..., 0)$.

[21] Even with two agents and a downward slopping Pareto frontier, as in Kocherlakota (1996), one may be interested in finding the efficient allocation that *ex-ante* gives the same utility to both agents. While this is trivial with our approach (just give the same initial weights in the **PP** problem), becomes very tricky with the GTW approach since the "right promise" must be made to determine the initial conditions.

[22] We refer to the *self-generation* and *factorization* properties of the operator characterizing incentive constraints (see, Abreu *et al.* (1990), and Chang (1998) and Ljungqvist and Sargent (2000) for macroeconomic applications).

the same principle applies to models with intertemporal constraints.

The strength of the APS approach, however, is in the study of models where one must know the set of feasible payoffs in order to characterize the efficient ones. In particular, as the work of Abreu (1988) shows, in order to characterize an efficient sub-game perfect equilibrium it is enough, in general, to know the *extremal points* of the set of feasible present value payoffs. More precisely, the *worst* equilibrium payoff may be sufficient to characterize the *best* equilibrium payoff, and a corresponding equilibrium path. Nevertheless, even in this class of models, it is often the case that one can separately obtain the worst sub-game perfect equilibrium payoff. If this is the case, such value plays the role of an intertemporal participation constraint in our framework. But, even when the computation of the *worst* equilibrium is not straightforward, our recursive approach, as a general method to obtain *extremal points,* may be a convenient approach[23]. In contrast, from a computational point of view, following the APS approach may not be very convenient general approach. In particular, while finding the set of feasible equilibrium payoffs may not be a hard computational problem when there is only one promised utility to consider, when there are n co-state variables a *set* in R^n has to be computed, which is not a straightforward computational problem. As we said, our approach is based on well specified initial values and deals with the computation of functions, which is a better understood problem[24].

Finally, an interesting -but not exclusive- feature of out approach is that the evolution of μ often helps to directly characterize the behavior of the model. For example, in models with participation constraints the μ's allow to interpret the behavior of the model as changing the Pareto weights sequentially depending on how binding the participation constraints become. In Ramsey type models the behavior of the μ's is associated with the commitment technology and the extent that markets are complete. For example, with full commitment and complete markets the μ's are constant after period one[25] Alternatively, in Marcet, Sargent and Seppala (1996) the behavior of μ characterizes optimal tax policies as a risk-adjusted martingale. In summary, having optimal policies dependent of an additional co-state vector μ is not just convenient from a technical point of view, but also provides a fairly transparent interpretation of how intertemporal incentive constraints affect (efficient) economic outcomes.

[23] *For example, Ljungqvist and Sargent (2000) show how separately compute the worst equilibrium in dynamic models with only one promised utility and no additional state variables. That our approach can also be used to obtain other extremal values may be seen by properly replacing the objective function, etc. Nevertheless we plan to make such application more explicit in future work.*

[24] For example, consider a version of the model of Chang (1998) with two agents. Certainly, an additional state variable needs to be introduced. If we use APS, the set of initial values would be a subset of R^2, and it would normally not be an interval, so it is difficult to characterize numerically. But using SPFE we know to set initial values for co-state variables to zero.

[25] For these models, our approach provides a recursive, alternative, formulation to the primal approach developed in Lucas and Stokey (1983) and Chari, *et al.* (1995).

7 Concluding remarks

We have shown that a large class of problems with implementability constraints can be analyzed by an equivalent recursive saddle point problem. This saddle point problem obeys a saddle point functional equation, which is a version of the Bellman equation. This approach works for a very large class of models with incentive constraints, restricted budget constraint, optimal policy, optimal regulation, etc. This means that a unified framework can be provided to analyze all these models. The key feature of our approach is that instead of having to write optimal contracts as history-dependent contracts one can write them as a stationary function of the standard state variables together with additional co-state variables. These co-state variables are -recursively- obtained from the Lagrange multipliers associated with the intertemporal incentive constraints, starting from pre-specified initial conditions. With such approach we aim to extends the existing set of tools available to study dynamic economies with intertemporal constraints.

Our current research aims at relaxing several of the assumptions, in particular that of full information, developing in detail some computational aspects of this method, and exploring a range of applications to several models, including strategic dynamic behavior, optimal policy and borrowing under incomplete insurance.

8 Authors' Affiliations

Albert Marcet, Institut d'Analisi Economica,
Bellaterra, Barcelona, Spain;
E-mail: albert.marcet@iae.csic.es
and
Ramon Marimon, European University Institute and Universitat Pompeu Fabra - CREi,,
Via delle Fontanelle 20, I-50014 San Domenico di Fiesole, Italy;
Telephone: +39-055-4685809, Fax: +39-055-4685894,
E-mail: ramon.marimon@eui.eu

APPENDIX

The proof of Proposition 1 relies on the following result:

Lemma 1A. Assume **A1-A6,**then

i) $\mathcal{B}(x,s)$ is non-empty, convex, bounded and $\sigma(\mathcal{L}_\infty,\, \mathcal{L}_1)$ closed; therefore it is $\sigma(\mathcal{L}_\infty,\, \mathcal{L}_1)$ compact

ii) Given $d \in R$, the set $\left\{ \boldsymbol{a} \in \mathcal{A} : f_{(x,\mu,s)}(\boldsymbol{a}) \geq d \right\}$ is convex and $\sigma(\mathcal{L}_\infty,\, \mathcal{L}_1)$ closed

The proof of Lemma 1A builds on three theorems. First, the *Urysohn metrization theorem* stating that regular topological spaces with a countable base are metrizable[26]. Second, the *Mackey-Arens theorem* stating that different topologies consistent with the same duality share the same closed convex sets; in our case, the duality is $(\mathcal{L}_\infty,\, \mathcal{L}_1)$ and the weakest and the strongest topology consistent with such duality; namely, the *weak-star*, $\sigma(\mathcal{L}_\infty,\, \mathcal{L}_1)$ and the Mackey $\tau(\mathcal{L}_\infty,\, \mathcal{L}_1)$. Third, the *Alaoglu theorem* stating that norm bounded $\sigma(\mathcal{L}_\infty,\, \mathcal{L}_1)$ closed subsets of $\mathcal{L}_\infty$ are $\sigma(\mathcal{L}_\infty,\, \mathcal{L}_1)$ compact[27].

Proof:

Assumptions **A2**, **A3** and **A5** imply that $\mathcal{B}(x,s)$ is convex, and closed under pointwise convergence. Since, by assumption **A1**, S is countable, *Urysohn metrization theorem* guarantees that $\mathcal{B}(x,s)$ is, in fact, $\sigma(\mathcal{L}_\infty,\, \mathcal{L}_1)$ closed. Assumptions **A4** and **A5** imply that $\mathcal{B}(x,s)$ is bounded in the $\|\cdot\|_\infty^\beta$ norm as needed for compactness, according to the *Alaoglu theorem.*

Assumptions **A3** and **A5** imply that $\mathcal{B}(x,s)$ and the upper contour sets $\{\boldsymbol{a} \in \mathcal{A} : f_\mu(\boldsymbol{a}) \geq d\}$, are convex and Mackey closed and, therefore, $\sigma(\mathcal{L}_\infty,\, \mathcal{L}_1)$ closed[28]■

Proof of Proposition 1: As in Bewley (1991), the central element of the proof follows from the *Hausdorff maximal principle* and an application of *the finite intersection property*[29]. Let $\mathcal{P}_d = \left\{ \boldsymbol{a} \in \mathcal{B}(x,s) : f_{(x,\mu.s)}(\boldsymbol{a}) \geq d \right\}$, then by Lemma 1 (see Appendix), $\mathcal{P}_d$ is $\sigma(\mathcal{L}_\infty,\, \mathcal{L}_1)$ closed and, for d low enough, it is non-empty. In fact, we can consider the family of sets $d \in D \subset R$ for which $\mathcal{P}_d \neq \varnothing$, $\{\mathcal{P}_d : d \in D\}$. The sets $\mathcal{P}_d$ are ordered by inclusion;

[26] See Dunford and Schwartz (1957) p. 24. in our case the metric we use is given by

$$\rho_\infty^\beta(\boldsymbol{a},\boldsymbol{b}) = \sum_{n=0}^{\infty} \beta^n \sup_{s^n \in S^n} \mid a_n(s) - b_n(s) \mid .$$

[27] See Schaefer (1966) p. 130 and p. 84, respectively.

[28] See Bewley (1972) for a proof of the Mackey continuity expected utility.

[29] See, Kelley (1955) p. 33-34. for the Hausdorff principle and the Minimal principle, and p. 136 for the theorem stating that "a set is compact if and only if every family of closed sets which has a the finite intersection property has a non-void intersection."

in fact, if $d' > d$ then $\mathcal{P}_{d'} \subset \mathcal{P}_d$ and every finite collection of them has a non-empty intersection (i.e., $\{\mathcal{P}_d : d \in D\}$ satisfies *the finite intersection property*), but then by compactness of $\mathcal{B}(x, s)$ any family of subsets of $\{\mathcal{P}_d : d \in D\}$ -say, $\{\mathcal{P}_d : d \in B \subset D\}$- has a non-empty intersection and, by inclusion, there is $\mathcal{P}_{\widehat{d}} = \cap \{\mathcal{P}_d : d \in B \subset D\} \neq \varnothing$. In particular, there is $\mathcal{P}_{d^*} = \cap \{\mathcal{P}_d : d \in D\} \neq \varnothing$ which -as the *the minimal principle* states- is a minimal member of the family $\{\mathcal{P}_d : d \in D\}$. It follows that if $\boldsymbol{a}^* \in \mathcal{P}_{d^*}$ then $f_{(x,\mu.s)}(\boldsymbol{a}^*) \geq f_{(x,\mu.s)}(\boldsymbol{a})$ for any $\boldsymbol{a} \in \mathcal{B}(x, s)$. Furthermore, if strictly concavity is assumed then $\mathcal{P}_{d^*}$ must be a singleton, otherwise convex combinations of elements of $\mathcal{P}_{d^*}$ will form a proper closed subset of $\mathcal{P}_{d^*}$ contradicting its minimality■

Proof of Theorem 1: It is an application of Theorem 1 (8.3) in Luenberger (1969), p. 217.

Lemma 2A. Let $W(x, \mu, s) \equiv V_\mu(x, s)$ be the value of $\mathbf{SPP}_\mu$ at (x, s), for an arbitrary $(x,\ \mu,\ s)$, then

i) $W(x, \cdot, s)$ is convex and homogeneous of degree one;

ii) if **A1**- **A5** are satisfied $W(\cdot, \mu, s)$ is continuous and uniformly bounded;

iii) if **A3** and **A6** are satisfied $W(\cdot, \mu, s)$ is concave, and

iv) if **A5**d is satisfied $W(\cdot, \mu, s)$ is differentiable at $(x,\ \mu,\ s)$, provided that $p(x, a^*(x, s), s) > 0$.

Proof: *i*) follows from the fact that, for any $\lambda > 0$, $f_{(x,\lambda\mu.s)}(\boldsymbol{a}) = \lambda f_{(x,\mu.s)}(\boldsymbol{a})$. Let $(\gamma^*, \boldsymbol{a}^*)$ satisfy

$$\begin{aligned} & f_{(x,\mu.s)}(\boldsymbol{a}^*) + \gamma g(\boldsymbol{a}^*)_0 \\ \leq \quad & f_{(x,\mu.s)}(\boldsymbol{a}^*) + \gamma^* g(\boldsymbol{a}^*)_0 \\ \leq \quad & f_{(x,\mu.s)}(\boldsymbol{a}) + \gamma^* g(\boldsymbol{a})_0, \end{aligned}$$

for any $\gamma \in R_+^{l+1}$ and $\boldsymbol{a} \in \mathcal{B}'(x, s)$, then $(\lambda\gamma^*, \boldsymbol{a}^*)$ satisfies

$$\begin{aligned} & f_{(x,\lambda\mu.s)}(\boldsymbol{a}^*) + \gamma g(\boldsymbol{a}^*)_0 \\ \leq \quad & f_{(x,\lambda\mu.s)}(\boldsymbol{a}^*) + \lambda\gamma^* g(\boldsymbol{a}^*)_0 \\ \leq \quad & f_{(x,\lambda\mu.s)}(\boldsymbol{a}) + \lambda\gamma^* g(\boldsymbol{a})_0, \end{aligned}$$

for any $\gamma \in R_+^{l+1}$ and $\boldsymbol{a} \in \mathcal{B}'(x, s)$. *ii*) and *iii*) are straightforward; in particular, *ii*) follows from applying the Theorem of the Maximum Stokey,, Lucas and Prescott,1989, Theorem 3.6) *iii*) follows from the fact that the constraint sets are convex and the objective function concave. Finally, *iv*) is an application of Benveniste and Scheinkman's theorem (Stokey, Lucas and Prescott, 1989, Theorem 4.10)■

Proof of Proposition 2: Given the assumptions of Proposition 2, for any (x, μ, s), and $\gamma \in \mathcal{R}_+^{l+1}$, let

$$F_{(x,\mu,s)}(\gamma) = \arg\sup_a \left\{\mu \left[h_0(x,\, a,\, s) + \beta \text{ E}\left[\chi^j \omega(x',\, \mu',\, s')|\, s\right]\right]\right\}$$

$$\text{s.t. } x' = \ell(x,\, a,\, s),\ p(x, a,\, s) \geq 0$$

$$\text{and } \mu' = \varphi(\mu,\, \gamma,\, s),$$

$$\text{and } h_1^j(x, a, s) + \beta \text{E}\left[\omega^j(x', \mu', s')|\, s\right] \geq 0,\ j = 0, ..., l \tag{38}$$

where $\chi^j = 1$, if $j = 0, ..., k, \chi^j = 0$, if $j = k+1, ..., l$. Since this is a standard maximization problem of a continuous function on a compact set, there is a solution $a^*(x, \mu, s; \gamma) \in F_{(x,\mu,s)}(\gamma)$. Furthermore, given that the constraint set is convex and has a non-empty interior (by **A2** and **A7b**), there is an associated multiplier vector; let $\gamma^{*j}(x, \mu, s; \gamma)$ be the multiplier corresponding to (38), for j. In particular, $\gamma^*(x, \mu, s; \gamma) \in G_{(x,\mu,s)}(a^*)$, where

$$\begin{aligned} G_{(x,\mu,s)}(a) &= \arg\inf_{\gamma \geq 0} \{\mu h_0(x,\, a,\, s) + \gamma h_1(x,\, a,\, s) + \beta \text{ E}\left[\varphi(\mu,\, \gamma,\, s)\omega(x',\, \varphi(\mu,\, \gamma,\, s),\, s')|\, s\right]\} \\ \text{s.t. } x' &= \ell(x,\, a,\, s),\ p(x, a,\, s) \geq 0 \end{aligned}$$

(see, for example, Luenberger (1969), p. 218). By homogeneity of degree zero of ω, $\gamma^* \in G_{(x,\mu,s)}(a^*)$ if, and only if, for all $\lambda > 0$, $\lambda\gamma^* \in G_{(x,\lambda\mu,s)}(a^*)$. Let

$$\begin{aligned} G^1_{(x,\mu,s)}(a) &= \arg\inf_{\{\gamma \geq 0 : \|\gamma\| \leq 1\}} \{\mu h_0(x,\, a,\, s) + \gamma h_1(x,\, a,\, s) + \beta \text{ E}\left[\varphi(\mu,\, \gamma,\, s)\omega(x',\, \varphi(\mu,\, \gamma,\, s),\, s')|\, s\right]\} \\ \text{s.t. } x' &= \ell(x,\, a,\, s),\ p(x, a,\, s) \geq 0, \end{aligned}$$

fix $\lambda \in (0, 1)$ and let

$$G^*_{(x,\mu,s)}(a) = \left\{\gamma^* \geq 0 : \gamma^* \in G^1_{(x,\mu,s)}(a) \text{ and } \lambda\gamma^* \in G^1_{(x,\lambda\mu,s)}(a)\right\}.$$

Notice that while $G^1_{(x,\mu,s)}(a)$ admits solutions which are not solutions to $G_{(x,\mu,s)}(a)$ – corresponding to a not satisfying (38) – $G^*_{(x,\mu,s)}(a) = G_{(x,\mu,s)}(a)$. The rest of the proof [details to be added] is an application of the *Theorem of the Maximum* (e.g. Stokey et al. (1989), p. 62) and of *Kakutani's Fixed Point Theorem* (e.g. Mas-Colell et al. (1995), p.953). By the former, $G^*_{(x,\mu,s)}(\cdot)$ and $F_{(x,\mu,s)}(\cdot)$ are upper hemicontinuous correspondences, non-empty and, by our assumptions on concavity and convexity, they are also convex valued, mapping a convex and (by **A2** and **A4**) compact set, $\left\{(a, \gamma) \in A \times \mathcal{R}_+^{l+1} : p(x, a,\, s) \geq 0\ \&\ \|\gamma\| \leq 1\right\}$, in itself; by *Kakutani's Fixed Point Theorem* there is a fixed point (a^*, γ^*) which is a solution to $\mathbf{DSPP}_{(x,\mu,s)}$. Furthermore, $F_{(x,\mu,s)}(\cdot)$ is a continuous function, when **A6s** is assumed■

Lemma 3A. M is a nonempty complete metric space.

Proof: That it is non-empty is trivial. Except for the homogeneity property, that every Cauchy sequence $\{\omega^n\} \in M$ converges to $\omega \in M$ satisfying $i)$ to $iii)$ follows from standard arguments (see, for example, Stokey, et al. (1989), Theorem 3.1 and Lemma 9.5). To see that the homogeneity property is also satisfied, for any (x, μ, s) and $\lambda > 0$,

$$\begin{aligned} & |\omega(x, \lambda\mu, s) - \omega(x, \mu, s)| \\ = \; & |\omega(x, \lambda\mu, s) - \omega^n(x, \lambda\mu, s) + \omega^n(x, \mu, s) - \omega(x, \mu, s)| \\ \leq \; & |\omega(x, \lambda\mu, s) - \omega^n(x, \lambda\mu, s)| + |W_n(x, \mu, s) - W(x, \mu, s)| \\ \to \; & 0 \end{aligned}$$

■

[The proofs of the remaining Lemmas have to be adapted to the new formulation of T !!]

Lemma 4A. The operator T maps M into itself.

Proof:

$$(T_K W)(x, \mu, s) = \mu h_0(x, a^*, s) + \gamma^* h_1(x, a^*, s) + \beta \mathrm{E} W(x^{*'}, \mu^{*'}, s')$$

therefore,

$$\begin{aligned} \|(T_K W)(x, \mu, s)\| \; \leq \; & \|\mu\| \, \|h_0(x, a^*, s)\| + \max\{1, \|\mu\|\} K \|h_1(x, a^*, s)\| \\ & + \beta \left\| W(x^{*'}, \mu^{*'}, s') \right\| \\ \leq \; & \|\mu\| \, \|h_0(x, a^*, s)\| + \max\{1, \|\mu\|\} K \|h_1(x, a^*, s)\| \\ & + \beta(\max\{1, \|\mu\|\} K + \|\mu\|) \left\| W(x^{*'}, \frac{\mu^{*'}}{\|\mu^{*'}\|}, s') \right\| \end{aligned}$$

It follows that the boundedness condition of $ii)$ is satisfied. A routine generalization of the Theorem of the Maximum (see, for example, Stokey, et al.,1989, Theorem 3.6) to this saddle point case, shows that $(TW)(\cdot, \cdot, s)$ is continuous. To see that the homogeneity properties are satisfied, let (a^*, γ^*) satisfy

$$(T_K W)(x, \mu, s) = h(x, a^*, \mu, \gamma^*, s) + \beta \mathrm{E} W(x^*, \mu^*, s')$$

then, for any $\lambda > 0$

$$\lambda (T_K W)(x, \mu, s) = \lambda[h(x, a^*, \mu, \gamma^*, s) + \beta \mathrm{E} W(x^*, \mu^*, s')]$$

Furthermore,

$$\begin{aligned} & h(x, a^*, \lambda\mu, \lambda\gamma^*, s) + \beta \mathrm{E} W(x^*, \lambda\mu^{*'}, s') \\ = \; & \lambda \left[h(x, a^*, \mu, \gamma^*, s) + \beta \mathrm{E} W(x^*, \mu^{*'}, s') \right] \end{aligned}$$

Now, let $\gamma \geq 0$, $\mu' = \varphi(\lambda\mu,\, \gamma,\, s')$ and $a \in A(x,s)$, $x' = \ell(x,\, a,\, s')$, then

$$\begin{aligned} & h(x,\, a^*,\, \lambda\mu, \gamma, s) \,+\, \beta \mathrm{E} W(x^*,\, \mu',\, s') \\ = \;& \lambda \left[h(x,\, a^*,\, \mu, \gamma\lambda^{-1}, s) \,+\, \beta \mathrm{E} W(x^*,\, \mu'\lambda^{-1},\, s') \right] \\ \geq \;& \lambda \left[h(x,\, a^*,\, \mu, \gamma^*, s) \,+\, \beta \mathrm{E} W(x^*,\, \mu^{*'},\, s') \right] \\ \geq \;& \lambda \left[h(x,\, a,\, \mu, \gamma^*, s) \,+\, \beta \mathrm{E} W(x',\, \mu^{*'},\, s') \right] \end{aligned}$$

It follows that,

$$\begin{aligned} (T_K W)(x,\, \lambda\mu,\, s) &= h(x,\, a^*,\;\; \lambda\mu,\;\; \lambda\gamma^*, s) \,+\, \beta \mathrm{E} W(x^*,\;\; \lambda\mu^{*'},\, s') \\ &= \lambda (T_K W)(x,\, \mu,\, s) \end{aligned}$$

[NOTE Show also concavity and convexity!!]■

Lemma 5A (monotonicity) Let $F, G \in M$ be such that $F \leq G$, then $(T_K F) \leq (T_K G)$.

Proof Fix $(\mu,\, x,\, s)$, then for any μ' satisfying $\mu' = \varphi(\mu,\, \gamma,\, s) \geq 0$,

$$\begin{aligned} & \max_{a \in A(x,s)} \{ h(x, a, \mu, \gamma,\, s) + \beta \mathrm{E} F(\ell(x,\, a,\, s), \mu',\, s') \} \\ \leq \;& \max_{a \in A(x,s)} \{ h(x, a, \mu, \gamma,\, s) + \beta \mathrm{E} G(\ell(x,\, a,\, s), \mu',\, s') \} \end{aligned}$$

It follows that

$$\begin{aligned} & \min_{\{\gamma \geq 0: \|\gamma\| \leq K_\mu\}} \max_{a \in A(x,s)} \{ h(x, a, \mu, \gamma,\, s) + \beta \mathrm{E} F(\ell(x,\, a,\, s), \varphi(\mu,\, \gamma,\, s),\, s') \} \\ \leq \;& \min_{\{\gamma \geq 0: \|\gamma\| \leq K_\mu\}} \max_{a \in A(x,s)} \{ h(x, a, \mu, \gamma,\, s) + \beta \mathrm{E} G(\ell(x,\, a,\, s), \varphi(\mu,\, \gamma,\, s),\, s') \} \end{aligned}$$

■

In our context, if $F \in M$ and $a \in \mathcal{R}$, we define the function $F + a \in M$ by $(F + a)(x,\, \mu,\, s) \,=\, F(x,\, \mu,\, s) \,+\, a$.

Lemma 6A (discounting) For all $W \in$ M , and $a \in \mathcal{R}_+$, $T_K(W \,+\, a) \leq T_K W \,+\, \beta a$.

Proof First notice that, for any $(x,\, \mu,\, s)$ and $\gamma \geq 0$,

$$\begin{aligned} & \max_{a \in A(x,s)} \{ h(x, a, \mu, \gamma,\, s) + \beta \mathrm{E}(W + a)(\ell(x,\, a,\, s), \varphi(\mu,\, \gamma,\, s),\, s') \} \\ = \;& \max_{a \in A(x,s)} \{ h(x, a, \mu, \gamma,\, s) + \beta \mathrm{E} W(\ell(x,\, a,\, s), \varphi(\mu,\, \gamma,\, s),\, s') + \beta a \} \\ = \;& \max_{a \in A(x,s)} \{ h(x, a, \mu, \gamma,\, s) + \beta \mathrm{E} W(\ell(x,\, a,\, s), \varphi(\mu,\, \gamma,\, s),\, s') \} + \beta a \end{aligned}$$

Now, using this equalities and the above definition for $F + a$,

$$T_K(W + a)(x, \mu, s)$$

$$\begin{aligned}
&= \min_{\{\gamma \geq 0 : \|\gamma\| \leq K_\mu\}} \max_{a \in A(x,s)} \{h(x, a, \mu, \gamma, s) + \beta \mathrm{E}(W + a)(\ell(x, a, s), \varphi(\mu, \gamma, s), s')\} \\
&= \min_{\{\gamma \geq 0 : \|\gamma\| \leq K_\mu\}} \max_{a \in A(x,s)} \{h(x, a, \mu, \gamma, s) + \beta \mathrm{E}W(\ell(x, a, s), \varphi(\mu, \gamma, s), s')\} + \beta a \\
&= (T_K W + \beta a)(x, \mu, s)
\end{aligned}$$

We have shown that $T_K(W + a) \leq T_K W + \beta a$■

Lemma 7A (Contraction property): The argument is standard. We show that the contraction property is satisfied. Let $F, G \in M$, then, using the homogeneity property of the functions in M, for any (x, μ, s),

$$\begin{aligned}
F(x, \mu, s) &= G(x, \mu, s) + [F(x, \mu, s) - G(x, \mu, s)] \\
&\leq G(x, \mu, s) + |F(x, \mu, s) - G(x, \mu, s)|
\end{aligned}$$

That is, $F \leq G + ||F - G||$. By the monotonicity and the discounting properties, it follows that $T_K F \leq T_K G + \beta ||F - G||$. But now, reversing the roles of F and G we obtain that

$$||T_K F - T_K G|| \leq \beta ||F - G||$$

Since $0 < \beta < 1$ we have that T_K is a contraction mapping■

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