“Uncertainty Averse Preferences”

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(2008)

Motivation/Goal

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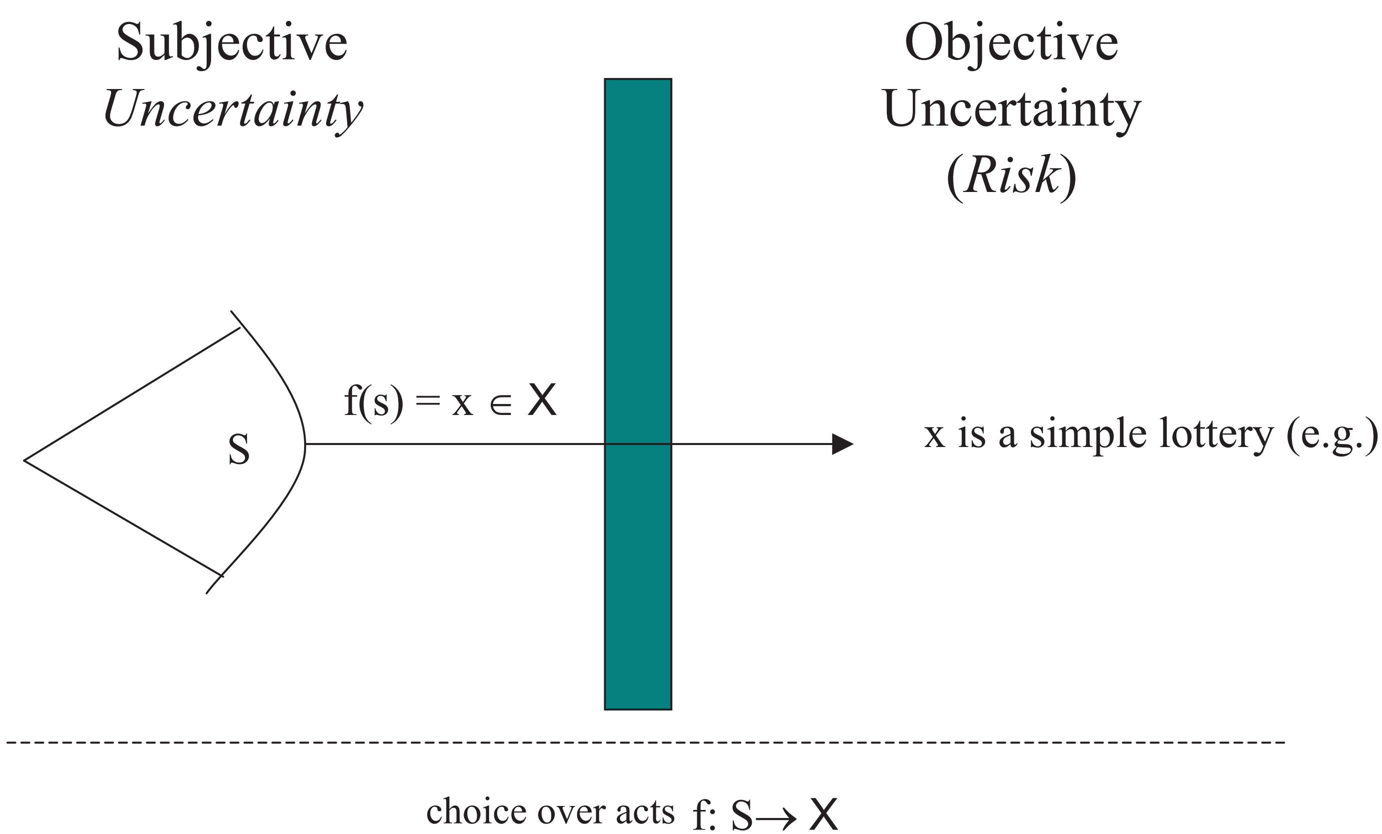
- Characterize general class of preferences which include, e.g.:
  1. Subjective Expected Utility
  $$U(f) = \int u(f)\, dp$$
  2. Multiple Priors Preferences
  $$U(f) = \min_{p \in M} \int u(f)\, dp$$
  3. Variational Preferences (VP)
  $$U(f) = \min_{p \in \Delta(S)} \left\{ \int u(f)\, dp + c(p) \right\}$$
  4. Smooth Preferences
  $$U(f) = \int \phi \left(\int u(f)\, dp \right) dm(p)$$

Motivation/Goal (cont'd)



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- Impose minimum number of cardinal and ordinal conditions:
  - (a) Full independence over (risky) lotteries
  - (b) Monotonicity
  - (c) Uncertainty aversion (ambiguity aversion)

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- Impose minimum number of cardinal and ordinal conditions:
  (a) Full independence over (risky) lotteries
  (b) Monotonicity
  (c) Uncertainty aversion (ambiguity aversion)
- Provide further results on quasiconcave duality

→ For instance, VP based on the Fenchel conjugate of a concave function.

→ $F(a) = \inf_{a^* \in A^*} \{\langle a^*, a \rangle - F^*(a^*)\}$

Outline

1. Setup
2. Preferences
3. Direct and Indirect Utility Function
4. Main Results

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- S = finite set of states of the world
- X = convex set of consequences (e.g., simple **lotteries**)
- Act $f : S \to X$, $f = \left(f(s_1), ..., f\left(s_{|S|}\right)\right)$
- $\mathcal{F} := X^S$ set of all possible acts

Preferences

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Axioms for UAP:

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A1. Complete Preference Relation

$$\succcurlyeq \text{ is a complete preorder with } \succ \neq \emptyset$$

A2. Monotonicity

$$f, g \in \mathcal{F} : f(s) \succcurlyeq g(s) \text{ for all } s \Rightarrow f \succcurlyeq g$$

A3. Uncertainty Aversion

$$f, g \in \mathcal{F}, \lambda \in (0,1) : f \sim g \Rightarrow \lambda f + (1-\lambda) g \succcurlyeq f$$

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A4. Risk Independence

$$x, y, z \in X, \lambda \in (0,1) : x \sim y \Rightarrow \lambda x + (1-\lambda) z \sim \lambda y + (1-\lambda) z$$

A5. Continuity $f, g, h \in \mathcal{F}$: $\{\lambda \in [0,1] : \lambda f + (1-\lambda) g \succcurlyeq h\}$ and $\{\lambda \in [0,1] : h \succcurlyeq \lambda f + (1-\lambda) g\}$ are closed

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Remark 1: A1 + A4 + A5 $\Rightarrow \exists\, u \in \mathbb{R}^X$ affine s.t., for all $x, y \in X$:
$x \succcurlyeq y$ iff $u(x) \geq u(y)$

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$\longrightarrow$ Now define utility $U : \mathcal{F} \rightarrow \mathbb{R}$ s.t. $U(f) = u(x_f)$.

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$\Diamond$ $\xi_f := u(f) = \left(u(f(s_1)), ..., u\left(f\left(s_{|S|}\right)\right)\right) \in u(X)^S \subseteq \mathbb{R}^S$

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Remark 3: Representation of $\trianglerighteq$: $I(\xi_f) = U(f)$. I is continuous, monotonic and quasiconcave.

Remark 4: I is a direct utility function on $u(X)^S$

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Fact:

$$V(b) = \min_{p \in \Delta(\{1,...,n\})} V_{ind}(p \cdot b, p)$$

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Theorem 9: $\succcurlyeq$ is UAP + A4 + A5 + 2 additional conditions iff

$\exists\ u \in \mathbb{R}^X$ affine, $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ such that for all $f, g \in \mathcal{F}$:

$$f \succcurlyeq g \text{ iff } \min_{p \in \Delta(S)} G\left(\int u(f)\, dp, p\right) \geq \min_{p \in \Delta(S)} G\left(\int u(g)\, dp, p\right)$$

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- G quasiconvex, $G(\cdot, p)$ increasing for all $p \in \Delta(S)$
- $\min_{p \in \Delta(S)} G(\iota, p) = \iota$
- Given u, G is unique an given by

$$G(\iota, p) = \sup_{f \in \mathcal{F}} \left\{ u(x_f) : \int u(f)\, dp = \iota \right\}$$

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Games against Nature: General Version

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$$\min_{p\in\Delta(S)} G\left(\int u(f)\,dp, p\right) = \min_{p\in\Delta(S)} \left\{\int u(f)\,dp + c\left(\int u(f)\,dp, p\right)\right\}$$

- Cost: $c\left(\int u(f)\,dp, p\right) = G\left(\int u(f)\,dp, p\right) - \int u(f)\,dp \geq 0$ (quasiconvex and grounded)
- Players' strategies:
  - DM $\to f$
  - Nature $\to p$

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Additively Separable: Variational Preferences

$$G(t,p) = \gamma(t) + c(p)$$

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Quasi-arithmetic:

$$\phi^{-1}\left(\int \phi(u(f))\, dq \right) = \min_{p \in \Delta^{\sigma}(q)} \left\{ \int u(f)\, dp + I_{\int u(f)dp}(p \| q) \right\}$$

$$\underbrace{I_{\int u(f)dp}(p\|q)}_{\text{statistical distance}} = \phi^{-1}\left(\inf_{k \geq 0} \left(k \int u(f)\, dp - \int \phi^{*}\left(k \frac{dp}{dq} \right) dq \right) \right) - \int u(f)\, dp$$

Main Results (cont'd)

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Multiplier preferences:

$$U(f) = \min_{p \in \Delta^{\sigma}(q)} \left\{ \int u(f)\, dp + \theta R(p \| q) \right\}$$

$$\phi(u) = \begin{cases} -e^{-\frac{1}{\theta} u}, \ \theta < \infty \\ u, \ \theta = \infty \end{cases}$$