“Nash equilibrium in games with incomplete preferences”

S. Bade (2005) Economic Theory

Motivation

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- Operational characterization (computation) of Nash equilibrium in normal form games with incomplete preferences

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  II. Tool for robust modelling

Outline

1. Definitions
2. General result
3. Vector-valued utility representation
4. Remarks on existence, and extensive form games
5. Cournot example

Definitions

Definitions

Normal form game:

$$G = \left\{ (A_i, \succsim_i)_{i \in I} \right\}$$

- $|I| < \infty$
- $A := \times_{i \in I} A_i$
- $\succsim_i =$ preference relation on A (transitive + reflexive = **preorder**)
- $N(G)$ = set of all NE of G

Definitions (cont'd)

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Optimization:

- $\succsim_i$ preorder on $A \longrightarrow$ maximum and maximal elements of A

  a) a^* maximum if $a^* \succsim_i a, \forall a \in A$

  b) a^* maximal if $\nexists\ a \in A$ such that $a \succ_i a^*$ (i.e., $a \succsim_i a^*$ but not $a^* \succsim_i a$)

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- Best response correspondence is $BR_i(a_{-i}) :=$ $\{a_i \in A_i : \nexists a_i' \in A_i \text{ such that } (a_i', a_{-i}) \succ_i (a_i, a_{-i})\}$

Definitions (cont'd)

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Nash equilibrium:

- Optimization + "Consistency"= NE

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Nash equilibrium:

- Optimization + "Consistency"= NE
- a^* is a NE strategy profile if $a^* \in \times_{i \in I} BR_i \left(a^*_{-i} \right)$

  (i.e., $\nexists\ i \in I$ such that $\left(a'_i, a^*_{-i} \right) \succ_i a^*$, some $a'_i \in A_i$)

Definitions (cont'd)

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Completion of a game:

- $\succcurlyeq_i'$ completes $\succcurlyeq_i$ (incomplete preference on A) if

  (i) $\succcurlyeq_i'$ is complete

  (ii) $a \succcurlyeq_i a' \Rightarrow a \succcurlyeq_i' a'$, and $a \succ_i a' \Rightarrow a \succ_i' a'$

Definitions (cont'd)

Completion of a game:

- $\succcurlyeq'_i$ completes $\succcurlyeq_i$ (incomplete preference on A) if
  (i) $\succcurlyeq'_i$ is complete
  (ii) $a \succcurlyeq_i a' \Rightarrow a \succcurlyeq'_i a'$, and $a \succ_i a' \Rightarrow a \succ'_i a'$
- $G' = \{(A_i, \succcurlyeq'_i)_{i \in I}\}$ is a completion of $G = \{(A_i, \succcurlyeq_i)_{i \in I}\}$
  if $\succcurlyeq'_i$ is a completion of $\succcurlyeq_i$, for each $i \in I$

General result

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- **Theorem 1**: Let $G = \left\{(A_i, \succcurlyeq_i)_{i \in I}\right\}$ be **any game**. Then

$$N(G) = \cup \{N(G') : G' \text{ is a completion of } G\}$$

Proof:

General result

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Proof:

"$\supseteq$" is easy. "$\subseteq$" follows from a version of Szpilrajn's theorem (uses Axiom of Choice, but not equivalent to)

General result (cont'd)

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- Pros:

  (1) Finding $N(G)$ = finding equilibria of a class of games with complete preferences

  (2) Players with complete preferences, but modeled with incomplete preferences $\rightarrow$ will find the "true" equilibrium (and many other as well)

General result (cont'd)

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(1) Finding $N(G)$ = finding equilibria of a class of games with complete preferences

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- Cons:

(1) Too general: how to construct the completions?

Vector-valued utility representation

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Objects:

Vector-valued utility representation

Objects:

- $\mathbf{u}^i : A \to \mathbb{R}^{m_i}$ represents $\succcurlyeq_i$: $a \succcurlyeq_i a'$ iff $\mathbf{u}^i(a) \geq \mathbf{u}^i(a')$
- Best response $BR_i(a_{-i}) :=$ $\left\{a_i \in A_i : \nexists a_i' \in A_i \text{ such that } \mathbf{u}^i(a_i', a_{-i}) > \mathbf{u}^i(a_i, a_{-i})\right\}$
- $G = \left\{\left(A_i, \mathbf{u}^i\right)_{i \in I}\right\}$ is the game $G = \left\{(A_i, \succcurlyeq_i)_{i \in I}\right\}$ where $\mathbf{u}^i$ represents $\succcurlyeq_i$

Vector-valued utility representation (cont'd)

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More Objects:

Vector-valued utility representation (cont'd)

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◇ $G_\beta := \left\{ \left(A_i, \beta^i \mathbf{u}^i \right)_{i \in I} \right\}$

$$\beta := \left(\beta^1, ..., \beta^{|I|} \right) \in \mathbb{R}^{\Sigma m_i}$$

$$\beta^i \mathbf{u}^i := \sum_{j=1}^{m_i} \beta^i_j u^i_j$$

Vector-valued utility representation (cont'd)

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- ◇ G_β is a linear completion of G if $\beta \in \Delta_+ := \Delta \cap \mathbb{R}^{\Sigma m_i}_{++}$, $\Delta := \times_{i \in I} \Delta^{m_i}$

Vector-valued utility representation (cont'd)

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- **Theorem 2**: Let $G = \left\{ \left(A_i, \mathbf{u}^i \right)_{i \in I} \right\}$, A_i is a nonempty and convex, and each u^i_j is concave in a_i. Then

$$\cup \{ N(G_\beta) : \beta \in \Delta_+ \} \subseteq N(G) \subseteq \cup \{ N(G_\beta) : \beta \in \Delta \}$$

Proof:

Vector-valued utility representation (cont'd)

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Proof:

First $\subseteq$ from Theorem 1. Second $\subseteq$ uses a separation hyperplane sort of argument (plus concavity)

Vector-valued utility representation (cont'd)

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- **Theorem 3**: Let $G = \left\{ \left(A_i, \mathbf{u}^i \right)_{i \in I} \right\}$ be a game, each A_i is nonempty and convex. Assume further that each u_j^i is strictly concave in a_i, for each $i \in I$. Then

$$N(G) = \cup \{ N(G_\beta) : \beta \in \Delta \}$$

Proof:

Vector-valued utility representation (cont'd)

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$$N(G) = \cup \{ N(G_\beta) : \beta \in \Delta \}$$

Proof:

Use strict concavity to show $\supseteq$

Remarks on existence, and extensive form games

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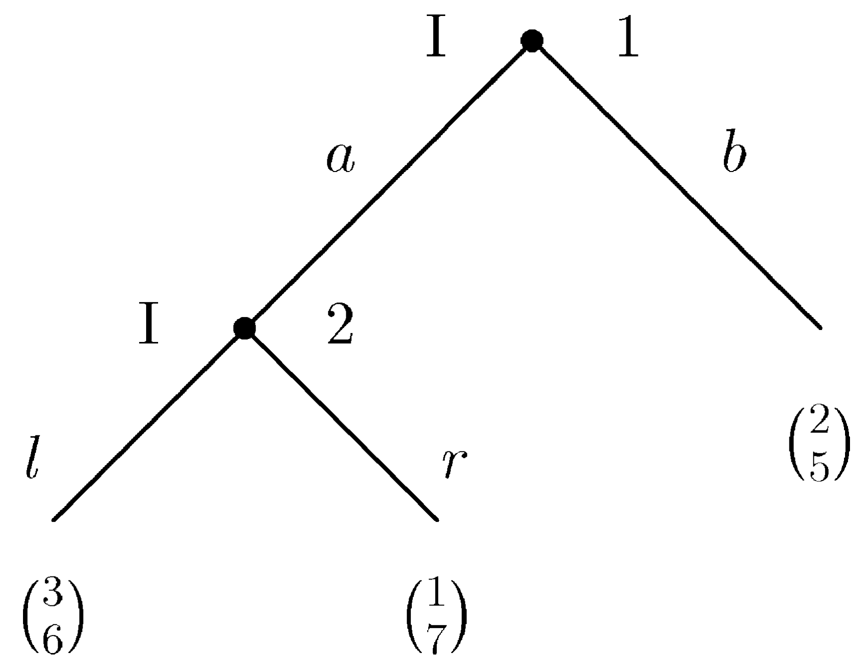
Extensive form games

- Cannot apply backward induction $\rightarrow$ e.g., hard to develop a general theory of repeated games with incomplete preferences (example next)

Remarks on existence, and extensive form games (cont'd)

- Example: $\sigma = (b, r)$ survives backward induction but is not NE (Krieger (2003), Math Meth Oper Res, Fig.2)



Cournot example

- Firms ($i = 1, 2$) have utility $\mathbf{u}^i(q) = (\text{profits}(q), \text{sales}(q))$, constant marginal cost $c = 0$, and face market demand $q^D = 1 - p$