“Expected Utility Theory without the Completeness Axiom”

Dubra, Maccheroni, and Ok (2004) JET

Motivation

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- Generalize the Expected Utility Theorem of vN-M to allow incompleteness of preferences

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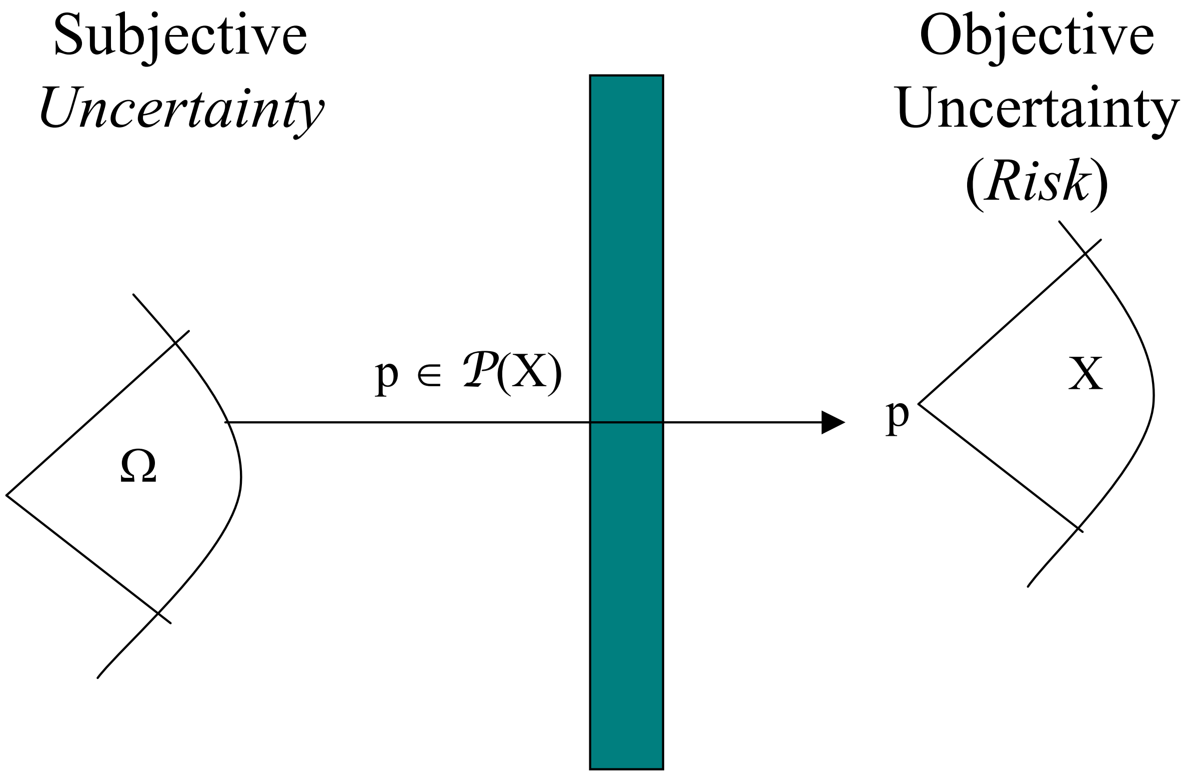
  II. Decision maker composed of several agents, each having a different objective function (multi-objective DM)



choice: Act h: $\Omega \rightarrow \mathcal{P}(X)$ Lottery p: $\mathcal{B} \rightarrow [0,1]$

(e.g., Bewley (1986))   (e.g., vN-M; Dubra-Maccheroni-Ok)

Outline

1. Definitions
2. Standard Expected Utility Theorem
3. Expected Multi-Utility Theorem
4. More

Definitions

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Assumption:

$\triangleright$ X is a compact metric space

Definitions (cont'd)

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Axioms:

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- Independence Axiom: For all $p, q, r \in \mathcal{P}(X)$, $\lambda \in (0,1]$:

$$p \succsim q \Rightarrow \lambda p + (1-\lambda) r \succsim \lambda q + (1-\lambda) r$$

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- Continuity Axiom: For any $(p_n) \to p$, $(q_n) \to q$ in $\mathcal{P}(X)$:

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◇ Remark 1: Independence Axiom + Continuity Axiom imply $p \succsim q \Leftrightarrow \lambda p + (1-\lambda) r \succsim \lambda q + (1-\lambda) r$

Standard Expected Utility Theorem

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- **Theorem 1** (vN-M) $\succcurlyeq$ is a complete preorder on $\mathcal{P}(X)$ satisfying the Independence + Continuity Axioms **iff** $\exists$ $u \in C(X) \subseteq \mathbb{R}^X$ such that, for all $p, q \in \mathcal{P}(X)$,

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◇ Remark 2: u is unique up to a positive affine transformation: any $(\alpha, \beta) \in \mathbb{R}_{++} \times \mathbb{R}$, $v = \alpha u + \beta$ also does the job

Expected Multi-Utility Theorem

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- **Theorem 2** $\succcurlyeq$ is a (possibly incomplete) preorder on $\mathcal{P}(X)$ satifying the Independence + Continuity Axioms **iff** $\exists$ closed and convex set $\mathcal{U} \subseteq C(X)$ such that for all $p, q \in \mathcal{P}(X)$,

$$p \succcurlyeq q \;\Leftrightarrow\; \int_X u dp \geq \int_X u dq, \text{ for all } u \in \mathcal{U}$$

Expected Multi-Utility Theorem (cont'd)

Expected Multi-Utility Theorem (cont'd)

◇ Remark 3

	Bewley	DMO
Choice	$f \in \mathcal{P}(X)^{\Omega}$	$p \in \mathcal{P}(X)$
Repres.	$f \succsim g$ iff $\mathbb{E}_{\mu}(\mathbb{E}_{f_\omega} u) \geq \mathbb{E}_{\mu}(\mathbb{E}_{g_\omega} u)$ all μ	$p \succcurlyeq q$ iff $\mathbb{E}_p u \geq \mathbb{E}_q u$ all u

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(1) Characterize $\succcurlyeq$ using a domination cone $\Rightarrow$ Shapley-Baucells Lemma: For all $p, q \in \mathcal{P}(X)$,

$$p \succcurlyeq q \;\Leftrightarrow\; p - q \in \mathcal{C}(\succcurlyeq),$$

where $\mathcal{C}(\succcurlyeq) := \cup_{\lambda > 0} \{\lambda (t - s) : t \succcurlyeq s, \text{ some } t, s \in \mathcal{P}(X)\}$

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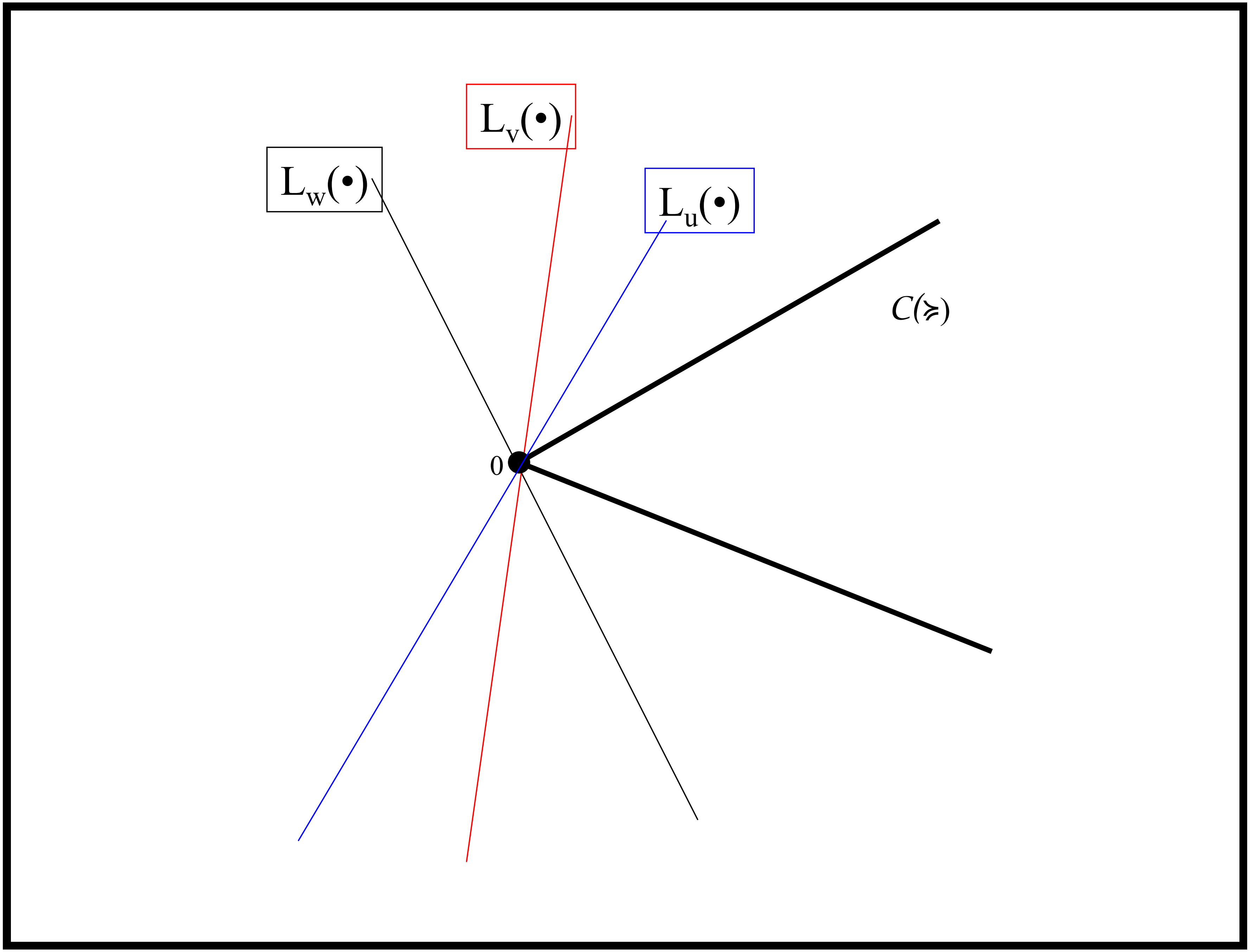
(3) $\mathcal{U} := \{u \in C(X) : L_u(\eta) \geq 0,$ all $\eta \in \mathcal{C}(\succcurlyeq)\}$, $L_u(\eta) := \int_X u d\eta$


$L_v(\bullet)$
$L_w(\bullet)$
$L_u(\bullet)$
$C(\succcurlyeq)$
0

Expected Multi-Utility Theorem (cont'd)

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- Uniqueness of the set $\mathcal{U}$ up to "positive affine transformations": $cl\left(cone\,(\mathcal{U}) + \{\theta 1_X\}_{\theta\in\mathbb{R}}\right)$ also does the job

More

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- Monetary prizes $X = [0, 1]$
- $\succcurlyeq$ also satisfies $p \succ_{FSD} q \Rightarrow p \succ q$

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- ◇ Monetary prizes $X = [0, 1]$
- ◇ $\succcurlyeq$ also satisfies $p \succ_{FSD} q \Rightarrow p \succ q$
- **Proposition** In the representation, each $u \in \mathcal{U}$ is weakly increasing and for all $0 \leq a < b \leq 1$, $u(a) < u(b)$ for some $u \in \mathcal{U}$