Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid Durable Consumption Goods

Grossman and Laroque, *Econometrica*, 1990

Presentation by Jonathan Halket, 4/24/07

$$V'(W) = u'(c)$$

$$EV'(W)(r_i - r_j) = 0$$

CCAPM rejected by data:

1. per capita consumption does not covary very much with stock returns
2. from envelope condition, same parameter must be used for intertemporal substitution and risk aversion.

Pref and Tech

Time is continuous.

$$E \int_0^\infty e^{-\delta t} \frac{K_t^a}{a} dt$$

Non-durables are in bulk (houses) and depreciate at rate α.

Proportional cost of selling house of size K: λ.
(Price of non-durables normalized to 1).

Risk free bond with risk-free rate r_f.

n risky assets whose returns follow: $\hat{\mu}_i dt + dw_{it}$.

Let B_t and $\underline{X}_t$ denote \$ amount of respective assets held by consumer at t.

$$Q_t = K_t + B_t + \underline{X}_t \cdot \underline{l}$$

Let τ denote a time which the consumer sells his house. In any dt where no sale occurs:

$$dQ_t = -\alpha K_t dt + r_f (Q_t - K_t) dt + \underline{X}_t \cdot d\underline{b}_t$$

If sale:

$$Q_\tau = Q_{\tau^-} - \lambda K_{\tau^-}$$

Consumer's problem

Given (Q_{0^-}, K_{0^-}), choose policy $(K_t, \underline{X}_t)$ (a fn of past values of $\underline{b}(t'), t' \leq t$) and stopping times τ, to max expected utility s.t.

- laws of motion for wealth
- no bankruptcy: $Q_t - \lambda K_t \geq 0$
- No ponzi schemes.

Let supremum be $V(Q, K)$

Note: $V(Q, K)$ homogenous of degree a, non-increasing in λ and bounded.

If he changes houses, he gets $\sup_c V(Q - \lambda K, c)$. If $V(Q, K) > \sup_c V(Q - \lambda K, c)$, he won't change.

$$V(Q,K) = \sup_{c,\tau,\underline{X}_t} E[\int_0^\tau e^{-\delta t}\frac{K_t^a}{a}dt + e^{-\delta\tau}V(Q_\tau - \lambda K_\tau, c)]$$

Change of variables:

$$y = \frac{Q}{K} - \lambda, \qquad \underline{x} = \frac{1}{K}\underline{X}$$

$$h(y) = K^{-a}V(Q,K) = V(\lambda + y, 1)$$

$$\bar{\delta} = \delta + a\alpha, \qquad r = \alpha + r_f$$

To get:

$$K^a h(y) = \sup_{c,\tau,\underline{x}_t} E[\int_0^\tau e^{-\delta t}\frac{(Ke^{-\alpha t})^a}{a}dt + e^{-\delta\tau}c^a h(\frac{Q_{\tau^-} - \lambda K_{\tau^-}}{c} - \lambda)]$$

$$M = \sup_c \left(\frac{Q_{\tau^-} - \lambda K_{\tau^-}}{c}\right)^{-a} h\left(\frac{Q_{\tau^-} - \lambda K_{\tau^-}}{c} - \lambda\right) = \sup_y (y+\lambda)^{-a} h(y)$$

$$h(y) = \sup_{\tau, \underline{x}_t} E\left[\int_0^\tau \frac{e^{-\bar{\delta} t}}{a} dt + e^{-\bar{\delta}\tau} M y_{\tau^-}^a\right]$$

s.t.

$$dy = \underline{x}_t \cdot d\underline{b} + r(y_t + \lambda - 1)dt$$

and $y_t \geq 0$

Note: M is bounded.

Leave aside issue of $\underline{x} = 0$.

Let M be exogenously given and let h(y; M) be solution to consumer's problem above. Then

- If $h(y) > y^a M$, then opt not to stop and $h(y)$ is twice continuously diff (except possibly at $y = 1 - \lambda$) and

$$\sup_{\underline{x}} [\frac{h''(y)}{2} \text{Var}dy + h'(y)Edy - \bar{\delta}h(y) + \frac{1}{a}] = 0$$

$$\text{Var}dy \equiv \underline{x}' \cdot \Sigma \cdot \underline{x}, \qquad Edy \equiv r(y + \lambda - 1) + \underline{x}'\underline{\mu}$$

- If $h(y) = y^a M$ then stop, h is cont. diff., and in the interior of the set $\{y|h(y) = y^a M\}$:

$$\sup_{\underline{x}} [\frac{h''(y)}{2} Vardy + h'(y)Edy - \bar{\delta}h(y) + \frac{1}{a}] \leq 0$$

Mean-Variance Portfolio

Where $h(y) > y^a M$ and h twice diff:

$$\underline{x}(y) = \frac{-h'(y)}{h''(y)} \Sigma^{-1} \cdot \underline{\mu}$$

$$\underline{X}(y) = s(y, K) \Sigma^{-1} \underline{\mu}$$

Sum over consumer (can be different $s()$):

$$\underline{p}_t = s_t \Sigma^{-1} \underline{\mu} \qquad \rightarrow \qquad s_t = \underline{p}_t' \Sigma \underline{p}_t / (\underline{p\mu})$$

which implies, if r_m is the market weighted portfolio:

$$\mu_i = \frac{\text{Cov}(\frac{d\hat{b}_i}{\hat{b}_i}, r_m)}{\text{Var}(r_m)} (Er_m - r_f)$$

Optimal Policy

If $M < W \quad \exists \quad y_1 \leq y^* \leq y_2$ such that $h(y)$ satisfies:

$$h(y) > My^a \qquad \text{iff} \qquad y \in (y_1, y_2)$$

and

$$M = (y^* + \lambda)^{-a} h(y^*) = \sup_y (y + \lambda)^{-a} h(y)$$

If $M \geq W$ then $h(y) = My^a$ (switch every period).

Smooth pasting condition helps here.

$$-\frac{1}{2}(\frac{\mu}{\sigma})^2\frac{(h'(y))^2}{h''(y)}+r(y+\lambda-1)h'(y)-\bar{\delta}h(y)+\frac{1}{a}=0 \qquad \text{for } y\in[y_1,y_2], y\neq 1-\lambda$$

$$h(y)\geq y^a M, \qquad \forall y$$

$$h(y_i)=y_i^a M$$

$$h'(y_i)=a y_i^{a-1}M$$

$$M=(y^*+\lambda)^{-a}h(y^*)=\sup_y (y+\lambda)^{-a}h(y)$$

Fix M. Guess y_1, get $h(y_1), h'(y_1)$ and use top to get $y_2 : h(y_2)=y_2^a M$. Check smooth pasting at y_2. If not, repeat.

Then check $y^*\in[y_1,y_2]$.

y_1 (y_2) is a str dec (inc) fn of λ.

λ	A	$(\bar{y}_1, \bar{y}^*, \bar{y}_2)$	$E\tau$ (yrs)	$\frac{E(X/Q)}{E(\tau)}$	$\frac{E(\frac{X}{Q-\lambda K})}{E(\tau)}$	P(buydown)	Rate of Wealth drift at y^*
.005	2.0	(.34,.58,.89)	11.47	.584	.587	.226	.0262
.25	2.0	(.4,.92,2.54)	27.42	.46	.604	.022	.0293
.05	2.0	(.29,.7,1.43)	28.94	.53	.57	.093	.0258
.05	1.75	(.27,.71,1.56)	27.33	.609	.654	.1	.0313
.05	1.5	(.26,.76,1.81)	25.64	.717	.768	.107	.0391
.05	1.1	(.26,1.08,3.16)	22.39	1.015	1.069	.121	.0622
.05	0.9	(.44,2.56,8.71)	20.46	1.303	1.335	.129	.0845
.005	0.9	(.76,2.03,4.38)	7.832	1.342	1.345	.261	.084
.25	0.9	(.39,3.62,18.3)	35.08	1.22	1.342	.042	.0875
.08	2.0	(.3,.74,1.65)	33.93	.511	.57	.069	.0262
.08	1.75	(.28,.76,1.82)	32.17	.587	.654	.074	.0317
.08	1.5	(.26,.81,2.12)	30.16	.693	.768	.082	.0397
.08	1.1	(.26,1.17,3.78)	26.49	.989	1.068	.095	.063
.08	0.9	(.4,2.78,10.56)	24.19	1.286	1.334	.103	.0849

Rise in λ increases avg time btw sales

λ	A	$(\bar{y}_1, \bar{y}^*, \bar{y}_2)$	$E\tau$ (yrs)	$\frac{E(X/Q)}{E(\tau)}$	$\frac{E(\frac{X}{Q-\lambda K})}{E(\tau)}$	P(buydown)	Rate of Wealth drift at y^*
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Wealth drifts up on average btw durable purchases

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Consumption changes infrequently: Cov(dk, db) small

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Avg. Wealth invested in risky falls when λ increases, but liquid wealth may not

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