On Efficient Distribution with Private Information

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Introduction

- Consider an exchange economy with individuals that experience idiosyncratic taste shocks which are their private information and a planner who looks for an efficient allocation of resources.
- In asymmetric information context: efficiency v. incentives
- Study the dynamics of the efficient distribution of consumption in an environment with private information on the individuals' side.
- Propose a recursive formulation of the problem

Model: Setup

- Two types of agents: planner and individuals
- Asymmetric information: idiosyncratic taste shocks of individuals are unobserved by planner
- Taste shocks $\theta \in \Theta \equiv \{\bar{\theta}_1, ..., \bar{\theta}_n\}$, $\bar{\theta}_1 > ... > \bar{\theta}_n$ and prob. measure μ with $\mu(\bar{\theta}_i) > 0\ \forall i$, and $E(\theta) = 1$.
- Shock history $\theta^t = (\theta_0, \theta_1, ..., \theta_t) \in \Theta^{t+1}$ and μ^{t+1} be the prob. measure of shock histories θ^t
- Individual's *reporting strategy*: $z = \{z_t(\theta^t)\}_{t=0}^{\infty} \in Z$ where $z_t : \Theta^{t+1} \to \Theta$
- Let $w \in D$ be the promised expected value to an individual.
- Individuals maximize lifetime utility given by

$$E\left\{\sum_{t=0}^{\infty}(1-\beta)\beta^t V(c_t(w, z^t))\theta_t\right\}$$

Model: Setup Cont'd

- A *plan* assigned by planner is a sequence $u = \{u_t(w, z^t)\}_{t=0}^{\infty} \in S$ such that:

$$\lim_{t \to \infty} \beta^t \sum_{s=0}^{\infty} \beta^s u_{t+s}(w, \theta^{t+s}) \theta_{t+s} = 0$$

- The individual's total expected utility function is given by:

$$U(w, u, z) = (1 - \beta) \sum_{t=0}^{\infty} \beta^t \int_{\Theta^{t+1}} u_t(w, z^t(\theta^t)) \theta_t d\mu^{t+1}$$

Planner's Problem

Choose $u = \{u_t(w, z^t)\}_{t=0}^{\infty} \in S$ to

$$\min \; y$$

subject to:

(i) Plan u is an *allocation*, i.e. u satisfies:

(a) $U(w, u, z^*) \geq U(w, u, z) \;\; \forall w \in D, \forall z \in Z$

where z^* is the truthful reporting strategy, i.e. $z_t^*(\theta^t) = \theta_t$

(b) $w = U(w, u, z^*) \;\; \forall w \in D$

(ii) Plan u attains the *utility distribution* ψ with resources y, i.e.

$$\int_{D \times \Theta^{t+1}} C[u_t(w, \theta^t)] d\mu^{t+1} d\psi \leq y \qquad \forall t$$

where $C : D \to \mathbb{R}_+$ is the inverse function of V

Recursive Formulation

- Consider $\sigma = \{f_t, g_t\}_{t=0}^{\infty}$ where $f_t, g_t : D \times \Theta \to D$

  The plan u generated by σ is derived from:

$$\begin{aligned} w_{t+1} &= g_t(w_t, z_t) \text{ with initial } w_0 \\ u_t(w_0, z^t) &= f_t[w_t(w_0, z^{t-1}), z_t] \end{aligned}$$

- A sequence σ is an *allocation rule* if:

  (i) $\lim_{t\to\infty} \beta^t w_t(w_0, \theta^{t-1}) = 0 \quad \forall w_0 \in D, \forall \{\theta_t\} \in \Theta^\infty$

  (ii) $w = \int_\Theta [(1-\beta) f_t(w,\theta)\theta + \beta g_t(w,\theta)] d\mu \quad \forall t, \forall w \in D$

  (iii) $(1-\beta) f_t(w,\theta)\theta + \beta g_t(w,\theta) \geq (1-\beta) f_t(w,\widehat{\theta})\theta + \beta g_t(w,\widehat{\theta})$

  $\forall t, \forall \theta, \widehat{\theta} \in \Theta$

Recursive Formulation Cont'd

Given an initial ψ, $\{g_t\}_{t=0}^{\infty}$ defines $\{\psi_t\}_{t=0}^{\infty}$ as $\psi_0 = \psi$ and $\psi_{t+1} = S_{g_t}\psi_t$ where the operator $S_g : M \to M$ is given by

$$(S_g\psi)(D_0) = \int_{B_g(D_0)} d\mu d\psi$$

where $B_g(D_0) = \{(w, \theta) \in D \times \Theta : g(w, \theta) \in D_0\}$ and D_0 is an arbitrary Borel subset of D

Results

Lemma 3.1.: Let $\psi \in M$ and suppose u attains ψ with resources y. Then there exists an allocation rule σ that attains ψ with resources y.

Lemma 3.2.: Let $\psi \in M$. Suppose σ attains ψ with resources y and that u is the utility plan generated by σ.
Then u is an allocation, and u attains ψ with resources y.

Bellman Equation for Efficient Allocations

Let $\varphi^*(\psi) = \inf\{y : \exists$ an alloc. u that attains ψ with resources $y\}$

An alloc. u is *efficient* if it attains a dist. ψ with resources $\varphi^*(\psi)$.

$$(T\varphi)(\psi) = \inf_{f,g \in B} \max \left\{ \int_{D\times\Theta} C[f(w,\theta)] d\mu d\psi, \quad \varphi(S_g \psi) \right\}$$

subject to:

$$(i)\ (1-\beta)f(w,\theta)\theta + \beta g(w,\theta) \geq (1-\beta)f(w,z)\theta + \beta g(w,z)$$

$$(ii)\ w = \int_\Theta [(1-\beta)f(w,\theta)\theta + \beta g(w,\theta)] d\mu \qquad \forall w \in D$$

Lemma 4.1.: φ^* is a fixed point of T.

Bellman Equation for Efficient Allocations Cont'd

Lemma 4.2.: Suppose there are functions φ_a, φ_c and φ such that $\forall \psi \in M$,

$$(i)\ \varphi_c \leq \varphi \leq \varphi_a$$

$$(ii)\ \lim_{n\to\infty} T^n \varphi_a = lim_{n\to\infty} T^n \varphi_c = \varphi.$$

Then $\varphi = \varphi^*$.

Candidate φ_a and φ_c are:

$$\varphi_a(\psi) = \int_D C(w) d\psi$$

$$\varphi_c(\psi) : \varphi_c(\psi) \text{ solves}$$

$$\min_u \int_{D\times\Theta} C[u(w,\theta)] d\psi d\mu$$

$$\text{subject to} \quad \int_\Theta \theta u(w,\theta) d\mu = w$$

Example: $V(c) = log(c)$ and $i.i.d.$ shocks

Bounding functions φ_a and φ_c are given by

$$\varphi_a = \int_D \exp(w) d\psi \qquad \alpha_a = 1$$

$$\varphi_c = \exp\{-E[\theta \log \theta]\} \int_D \exp(w) d\psi \qquad \alpha_c = \exp\{-E[\theta \log \theta]\}$$

Consider the following *Problem P*

$$\phi(\alpha) = \min_{r,h} \max \left\{ \int_\Theta \exp(r(\theta)) d\mu, \quad \alpha \int_\Theta \exp(h(\theta)) d\mu \right\}$$

subject to: $(1-\beta)\theta r(\theta) + \beta h(\theta) \geq (1-\beta)\theta r(z) + \beta h(z)$

$$\int_\Theta \left[(1-\beta)\theta r(\theta) + \beta h(\theta)\right] d\mu = 0$$

Example: $V(c) = log(c)$ and $i.i.d.$ shocks Cont'd

Lemma 5.2.: For any $\alpha > 0$ let the minimum of *Problem P* attained by $r(\theta;\alpha)$, $h(\theta;\alpha)$. Let $\varphi(\psi) = \alpha \int_D \exp(w) d\psi$.Then,

$$(T\varphi)(\psi) = \phi(\alpha) \int_D \exp(w) d\psi$$

and the optimal pair (f, g) is $(r(\theta;\alpha) + w,\ h(\theta;\alpha) + w)$.

Example: $V(c) = log(c)$ and $i.i.d.$ shocks Cont'd

The evolution of individual promised values and period utility are respectively:

$$w_t(w_0, \theta^{t-1}) = w_0 + \sum_{s=0}^{t-1} h(\theta_s)$$

$$u_t(w_0, \theta^t) = f_t(w_t(w_0, \theta^{t-1}), \theta_t) = w_0 + \sum_{s=0}^{t-1} h(\theta_s) + r(\theta_t)$$

The variance of the cross-sectional distribution of promised values and period utilities are:

$$Var(w_t(w_0, \theta^{t-1})) = Var(w_0) + t \cdot Var(h(\theta))$$

$$Var(u_t(w_0, \theta^t)) = Var(w_0) + t \cdot Var(h(\theta)) + Var(r(\theta))$$