Beliefs and Private Monitoring

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Introduction

- Finding equilibria in repeated games with private monitoring is known to be hard.
- Infinite number of histories where incentives have to be checked, and for each case beliefs have to be computed, which is a *"difficult (if not impossible) task"*.
- No recursive formulation for sequential equilibria in repeated games with private monitoring.
- For an important subclass of strategies, this paper provides readily checkable and computable necessary and sufficient conditions for equilibrium.
- Develops new, recursive set based methods to compute equilibria.

Model: Repeated Game Γ^∞

- N players: $i = 1, ..., N$
- Individual actions $a_i \in A_i$ (finite) and action profile $a = (a_1, ..., a_N)$
- Information structure:
  - At beginning of game, each player receives private signal $s_i \in S_i$ where $S = S_1 \times ... \times S_N$ is finite and $x(s)$ is the probability of $s \in S$.
  - Every period player i observes private outcome $y_i \in Y_i$ (finite) where $y = (y_1, ..., y_N)$ occurs with probability $P(y|a) > 0 \;\; \forall (a, y)$ *(full support)*

    Let $h_{i,t} = (a_{i,t}, y_{i,t}) \in H_i = A_i \times Y_i$ and $h_i^t = (h_{i,0}, ..., h_{i,t-1})$.
- Payoffs: Period utility for player i is $u_i : H_i \to \mathbb{R}$

  Expected lifetime utility is given by $(1 - \beta) E \sum_{t=0}^{\infty} \beta^t u_{i,t}$

Let a *(behavior) strategy* for player i be given by $\sigma_i = \{\sigma_{i,t}\}_{t=0}^{\infty}$ such that $\sigma_{i,t} : S_i \times H_{i,t-1} \to \Delta(A_i)$

Finite State Strategies

A *finite state strategy* (or *finite automaton*) is defined by:

(i) Private state space Ω_i (with D_i elements ω_i)

(ii) Function $p_i(a_i|\omega_i)$

(iii) Deterministic transition fcn. $\omega_i^+ : \Omega_i \times H_i \rightarrow \Omega_i$ where $H_i = A_i \times Y_i$

(iv) Deterministic mapping $\omega_i^0 : S_i \rightarrow \Omega_i$

Using this automaton representation the strategy is defined recursively in the private states

$$\begin{aligned} \sigma_{i,0}(s_i)(a_i) &= p_i(a_i|\omega_i^0(s_i)) \\ \sigma_{i,t}(\omega_{i,t})(a_i) &= p_i(a_i|\omega_{i,t}) \\ \omega_{i,t} &= \omega_i^+(\omega_{i,t-1}, a_i, y_i) \end{aligned}$$

Beliefs: Bayes' Rule Updating

Player i's belief over initial states of opponents, $\omega_{-i,0}$, are given by

$$m_i^0(s_i)(\omega_{-i,0}) = \sum_{s_{-i} \text{ s.t. } \omega^0_{-i}(s_{-i})=\omega_{-i,0}} \frac{x(s_i, s_{-i})}{\sum_{\bar{s}_{-i}} x(s_i, \bar{s}_{-i})}$$

Player i's belief over $\omega_{-i,t}$, if beliefs over $\omega_{-i,t-1}$ were $m_i \in \Delta^{D_{-i}}$ and observed $h_i = (a_i, y_i)$ are given by

$$B_i(m_i, h_i|\sigma_{-i})(\omega'_{-i}) = \frac{\sum_{\omega_{-i}} m_i(\omega_{-i}) H_i(\omega_{-i}, \omega'_{-i}, h_i|\sigma_{-i})}{\sum_{\omega_{-i}} m_i(\omega_{-i}) F_i(\omega_{-i}, h_i|\sigma_{-i})}$$

where

$$\begin{aligned}
F_i(\omega_{-i}, h_i|\sigma_{-i}) &= \sum_{(a_{-i}, y_{-i})} p_{-i}(a_{-i}|\omega_{-i}) P(y_i, y_{-i}|a_i, a_{-i}) \\
H_i(\omega_{-i}, \omega'_{-i}, h_i|\sigma_{-i}) &= \sum_{h_{-i} \in G_{-i}(\omega_{-i}, \omega'_{-i}|\sigma_{-i})} p_{-i}(a_{-i}|\omega_{-i}) P(y_i, y_{-i}|a_i, a_{-i}) \\
G_{-i}(\omega_{-i}, \omega'_{-i}|\sigma_{-i}) &= \left\{ h_{-i} = (a_{-i}, y_{-i}) | \omega^+_{-i}(\omega_{-i}, a_{-i}, y_{-i}) = \omega'_{-i} \right\}
\end{aligned}$$

Beliefs: Bayes' Rule Updating Cont'd

Let

$$\begin{aligned} B_i^1(m_i, h_i|\sigma_{-i}) &= B_i(m_i, h_i|\sigma_{-i}) \\ B_i^s(m_i, h_i^s|\sigma_{-i}) &= B_i(B_i^{s-1}(m_i, h_i^{s-1}|\sigma_{-i}), h_{i,s-1}|\sigma_{-i}) \qquad \forall s \geq 2 \end{aligned}$$

Then, let player i's beliefs over ω_{-i}, conditional on observing s_i and h_i^t, be given by

$$\mu_i(s_i, h_i^t) = B_i^t(m_i^0(s_i), h_i^t|\sigma_{-i})$$

Let

$$Ev_i(\omega_i(s_i, h_i^t), \mu_i(s_i, h_i^t)|\sigma) = \sum_{\omega_{-i}} \mu_i(s_i, h_i^t)(\omega_{-i}) v_i(\omega_i(s_i, h_i^t), \omega_{-i}|\sigma)$$

Equilibrium Concept

DEF: A correlation device x and finite state strategy σ (with beliefs μ) form a *Correlated Sequential Equilibrium* (CSE) of Γ^{∞} if $\forall i, t, s_i, h_i^t$, and arbitrary $\widehat{\sigma}_i$,

$$Ev_i(\omega_i(s_i, h_i^t), \mu_i(s_i, h_i^t)|\sigma) \geq EV_{i,t}(s_i, h_i^t, \mu_i(s_i, h_i^t)|\widehat{\sigma}_i, \sigma_{-i})$$

One-shot Deviation Principle: Suppose a correlation device and a finite state strategy σ satisfy $\forall i, t, s_i, h_i^t$, and arbitrary $\widehat{a}_i$,

$$Ev_i(\omega_i(s_i, h_i^t), \mu_i(s_i, h_i^t)|\sigma) \geq \sum_{\omega_{-i}} \mu_i(s_i, h_i^t)(\omega_{-i})\{\sum_{a_{-i}} p_{-i}(a_{-i}|\omega_{-i}) \sum_{y} P(y|\widehat{a}_i, a_{-i})$$
$$[(1-\beta)u_i(\widehat{a}_i, y_i) + \beta v_i(\omega_i^+(\omega_i(s_i, h_i^t), \widehat{a}_i, y_i), \omega_{-i}^+(\omega_{-i}, a_{-i}, y_{-i})|\sigma)]\}$$

Set Based Methods

Let $M_i(\omega_i) \subseteq \Delta^{D-i}$ be a closed, convex set of beliefs.
Let M_i be a collection of D_i sets, $M_i(\omega_i)$, one for each ω_i. and let $\mathbf{M}$ be the space of M_i.

Consider $T : \mathbf{M} \rightarrow \mathbf{M}$ such that

$$T(M_i) = \left\{ T(M_i)(\omega_i') : \omega_i' \in \Omega_i \right\}$$

where

$$T(M_i)(\omega_i') = co(\{m_i' | \exists \omega_i \in \Omega_i, m_i \in M_i(\omega_i) \text{ and } (a_i, y_i) \in G_i(\omega_i, \omega_i' | \sigma_i) \text{ such that } m_i' = B_i(m_i, a_i, y_i | \sigma_{-i})\})$$

Let $T^1(M_i) \equiv T(M_i)$ and for $n \geq 2$, $T^n(M_i) \equiv T(T^{n-1}(M_i))$.

Fixed Point of Operator T

- T is a monotone operator, i.e. if $M_i^0 \subseteq M_i^1$, then $T(M_i^0) \subseteq T(M_i^1)$
- By Tarski's Fixed Point theorem, $\exists$ a unique greatest FP, which we denote $\overline{M}_i$.

Let $\overline{\Delta}_i$ denote the collection of D_i D_i-dimensional unit simplexes. Since $T(\overline{\Delta}_i) \subseteq \overline{\Delta}_i$, then $\lim_{n\to\infty} T^n(\overline{\Delta}_i) = \overline{M}_i$.

Let π be the invariant distribution for a finite state Markov chain given by:

$$\tau(\omega, \omega'|\Omega, p, \omega^+) = \sum_{(a,y) \text{ s.t. } (a_i,y_i)\in G(\omega_i,\omega_i')} P(y|a)\Pi_{i=1}^N p_i(a_i|\omega_i)$$

Consider $S = \Omega$, $x = \pi$, $\omega_i^0(s_i = \omega_i) = \omega_i$. Let $M_{\pi,i}(\omega_i) \equiv \{m_i^0(s_i = \omega_i)\}$.

Lemma: $\forall i$, $M_{\pi,i} \subseteq T(M_{\pi,i})$

Sufficient Conditions

Theorem 1: Consider behaviors for players $i = 1, ..., N$ described by a state space $\Omega = \Omega_1 \times ... \times \Omega_N$, action probs. $p = \{p_i(a_i|\omega_i)\}_{i=1}^N$ and transition fcns. $\omega^+ = \{\omega_i^+(\omega_i, a_i, y_i)\}_{i=1}^N$. **If**

$$Ev_i(\omega_i, m_i|\sigma) \geq \sum_{\omega_{-i}} m_i(\omega_{-i})\{\sum_{a_{-i}} p_{-i}(a_{-i}|\omega_{-i}) \sum_{y} P(y|\widehat{a}_i, a_{-i})$$
$$[(1-\beta)u_i(\widehat{a}_i, y_i) + \beta v_i(\omega_i^+(\omega_i, \widehat{a}_i, y_i), \omega_{-i}^+(\omega_{-i}, a_{-i}, y_{-i})|\sigma)]\}$$

$\forall i, \widehat{a}_i, \omega_i$ and m_i such that:

(a) m_i is an extreme point of a set $M_i(\omega_i)$ such that $\overline{M}_i(\omega_i) \subseteq M_i(\omega_i)$

(b) m_i is an extreme point of $M^*_{\pi,i}(\omega_i)$ where $M^*_{\pi,i} \equiv \lim_{n\to\infty} T^n(M_{\pi,i})$.

Then, $\exists$ starting conditions (signal space S, probs. $x(s)$, fcns. ω^0) such that x and σ defined by $(\Omega, p, \omega^+, \omega_0)$ form a CSE

Necessary Conditions

Theorem 2: Consider behaviors for players $i = 1, ..., N$ described by a state space $\Omega = \Omega_1 \times ... \times \Omega_N$, action probs. $p = \{p_i(a_i|\omega_i)\}_{i=1}^N$ and transition fcns. $\omega^+ = \{\omega_i^+(\omega_i, a_i, y_i)\}_{i=1}^N$.
Suppose $\forall i$ and $\emptyset \neq M_i \in \mathbf{M}$, $\lim_{n\to\infty} T^n(M_i) = \overline{M}_i$.

Then, $\exists$ starting conditions (signal space S, probs. $x(s)$, fcns. ω^0) such that x and σ defined by $(\Omega, p, \omega^+, \omega_0)$ form a CSE **only if**

$$Ev_i(\omega_i, m_i|\sigma) \geq \sum_{\omega_{-i}} m_i(\omega_{-i})\{\sum_{a_{-i}} p_{-i}(a_{-i}|\omega_{-i}) \sum_y P(y|\widehat{a}_i, a_{-i})$$
$$[(1-\beta)u_i(\widehat{a}_i, y_i) + \beta v_i(\omega_i^+(\omega_i, \widehat{a}_i, y_i), \omega_{-i}^+(\omega_{-i}, a_{-i}, y_{-i})|\sigma)]\}$$

What conditions ensure $\lim_{n\to\infty} T^n(M_i) = \overline{M}_i \ \forall \emptyset \neq M_i \in \mathbf{M}$?

Assumption: Strategy profile σ is *connected*, i.e. $\exists\ L : \forall \omega_i^0, \omega_i^1 \in \overline{\Omega}_i$, we can transit from $\omega_{i,0} = \omega_i^0$ to $\omega_{i,L} = \omega_i^1$ in L steps.

Example

Consider the following repeated game:

- Each players $i \in \{1, 2\}$ can take action $a_i \in \{C, D\}$
- Each player can obtain a private outcome $y_i \in \{G, B\}$ and $P(y|a)$ is such that if m players play C

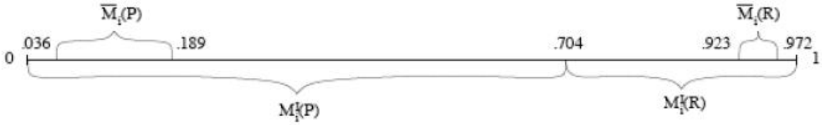
Output	Probability
GG	$p_m(1-\varepsilon)^2 + (1-p_m)\varepsilon^2$
GB	$p_m(1-\varepsilon)\varepsilon + (1-p_m)(1-\varepsilon)\varepsilon$
BG	$p_m(1-\varepsilon)\varepsilon + (1-p_m)(1-\varepsilon)\varepsilon$
BB	$p_m\varepsilon^2 + (1-p_m)(1-\varepsilon)^2$

- Parameters: $\beta = 0.9$, $p_0 = 0.3$, $p_1 = 0.55$, $p_2 = 0.9$
- Payoffs: $u_i(a_i, y_i) = 1_{y_i = G} - 0.4 \cdot 1_{a_i = C}$

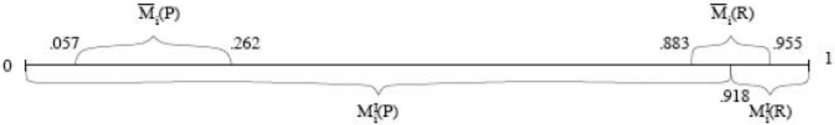
Strategy: Tit-For-Tat

- $\Omega_i = \{R, P\}$
- $p_i(C|R) = 1,\ p_i(D|P) = 1$
- $\omega_i^+(\omega_i, a_i, G) = R,\ \omega_i^+(\omega_i, a_i, B) = P$
- Since $D_{-i} = 2$, $\overline{M}_i(\omega_i)$ is simply a closed interval specifying range of probs. player $-i$ is in R, given player i is in state ω_i.
  - Compute $\overline{M}_i(R)$ and $\overline{M}_i(P)$ as $\lim_{n\to\infty} T^n(M_i) = \overline{M}_i$ for any $M_i \neq \emptyset$.
  - Check incentive constraints using as beliefs extreme points of $\overline{M}_i(\omega_i)$ for $\omega_i \in \{R, P\}$
  - If ICs hold, Theorem 1 delivers one starting condition: $S = \Omega$, $x = \pi$ where π is the invariant distribution of joint states ω, and $\omega_i^0(s_i = \omega_i) = \omega_i$.

Strategy: Tit-For-Tat



Tit–for–Tat, $\varepsilon = 0.025$



Tit–for–Tat, $\varepsilon = 0.04$

Concluding Remarks

- Set based methods can be applied to broader set of strategies
- If ICs hold strictly, this CSE is robust to small perturbations of:
  - Stage game payoffs or discount factors
  - Monitoring technology (function $P(y|a)$)