Recursive Contracts

Albert Marcet and Ramon Marimon (1998)

Håkon Tretvoll

New York University

January 29, 2008

Introduction

- Key condition in standard dynamic programming: only past variables influence set of feasible current actions
- Many dynamic models fail to satisfy this
  $\Rightarrow$ Bellman equation fails to hold
  (Ex: Models with participation constraints and Ramsey problems)
- MM (1998): for many problems that are not recursive, since constraints depend on future variables, an equivalent problem with a recursive formulation can be constructed

Planner's Problem (PP, Program 1)

$$
\begin{aligned}
\sup_{\{a_t\}} \quad & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t r(x_t, a_t, s_t) \\
\text{s.t.} \quad & x_{t+1} = \ell(x_t, a_t, s_{t+1}), \quad a_t \in A(x_t, s_t), \qquad (1a)
\end{aligned}
$$

x_0, s_0 given.

Planner's Problem (PP, Program 1)

◂ Example (1) ◂ Example (2)

$$\sup_{\{a_t\}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t r(x_t, a_t, s_t)$$

$$\text{s.t.} \quad x_{t+1} = \ell(x_t, a_t, s_{t+1}), \quad a_t \in A(x_t, s_t), \qquad (1a)$$

$$g_1^j(x_t, a_t, s_t) + \mathbb{E}_t \sum_{n=1}^{N_j} \beta^n g_2^j(x_{t+n}, a_{t+n}, s_{t+n}) \geq 0,$$

$$j = 1, \ldots, k; t \geq 0 \qquad (2)$$

$$x_0, s_0 \text{ given.}$$

Notation

- r: return function
- β: discount factor ($\in (0, 1)$)
- $\{s_t\}$: exogenous Markov process
- x: endogenous state variable
- a: control variable, subject to constraint A
- ℓ: transition function
- r, β, A, ℓ and the transition of s_t assumed known
- MM (1998) consider $N_j = 1$ or $N_j = \infty$

Constraint (2) $\Rightarrow$ standard d.p. inapplicable

- Without constraint (2) $\Rightarrow$ standard dynamic programming $\Rightarrow$ solution $a_t = f(x_t, s_t)$
- (2) involves expected values of future variables $\Rightarrow$ solution is *not* of the form $a_t = f(x_t, s_t)$
- **Contribution:** MM (1998) show how to convert PP into a recursive saddle point problem (SPP), and then extend dynamic programming theory to such problems

Converting PP $\rightarrow$ SPP

Consider case $j = 1$ and $N = \infty$

- Step 1: Form Lagrangian w.r.t. constraint (2):

$$L \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big[r(x_t, a_t, s_t) + \gamma_t \left(g_1(x_t, a_t, s_t) + \mathbb{E}_t \sum_{n=1}^{\infty} \beta^n g_2(x_{t+n}, a_{t+n}, s_{t+n}) \right) \Big]$$

subject to (1a), and given $\gamma_t \geq 0$.

Converting PP $\rightarrow$ SPP

Introduce co-state variable μ_t

- Step 2: Key algebraic step:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \gamma_t \mathbb{E}_t \sum_{n=1}^{\infty} \beta^n g_2(x_{t+n}, a_{t+n}, s_{t+n}) = \mathbb{E}_0 \sum_{t=1}^{\infty} \beta^t \mu_t g_2(x_t, a_t, s_t)$$

where $\mu_0 = 0$ and, for all $t \geq 0, \mu_{t+1} = \mu_t + \gamma_t$

Converting PP $\rightarrow$ SPP

Introduce co-state variable μ_t

- Step 2: Key algebraic step:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \gamma_t \mathbb{E}_t \sum_{n=1}^{\infty} \beta^n g_2(x_{t+n}, a_{t+n}, s_{t+n}) = \mathbb{E}_0 \sum_{t=1}^{\infty} \beta^t \mu_t g_2(x_t, a_t, s_t)$$

where $\mu_0 = 0$ and, for all $t \geq 0, \mu_{t+1} = \mu_t + \gamma_t$

$$\Rightarrow L \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_t [r(x_t, a_t, s_t) + \gamma_t g_1(x_t, a_t, s_t) + \mu_t g_2(x_t, a_t, s_t)]$$

Converting PP $\rightarrow$ SPP

- Step 3: Define the functions h and φ as:

$$\begin{aligned} h(x, a, \mu, \gamma, s) &\equiv r(x, a, s) + \gamma g_1(x, a, s) + \mu g_2(x, a, s) \\ \varphi(\mu, \gamma, s) &\equiv \mu + \gamma \end{aligned}$$

Recursive Saddle Point Problem (SPP, Program 2)

◂ Example

$$\inf_{\{\gamma_t\}} \sup_{\{a_t\}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t h(x_t, a_t, \mu_t, \gamma_t, s_t)$$

$$\text{s.t.} \quad x_{t+1} = \ell(x_t, a_t, s_{t+1}), \quad a_t \in A(x_t, s_t), \quad (1a)$$

$$\mu_{t+1} = \varphi(\mu_t, \gamma_t, s_{t+1}), \quad \gamma_t \geq 0, t \geq 0 \quad (3)$$

$$\mu_0 = 0, \qquad x_0, s_0 \text{ given},$$

where the mappings h and φ are derived from r, g and N_j.

Recursive Saddle Point Problem (SPP, Program 2)

◂ Example

$$\inf_{\{\gamma_t\}} \sup_{\{a_t\}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t h(x_t, a_t, \mu_t, \gamma_t, s_t)$$

$$\text{s.t.} \quad x_{t+1} = \ell(x_t, a_t, s_{t+1}), \quad a_t \in A(x_t, s_t), \quad (1a)$$

$$\mu_{t+1} = \varphi(\mu_t, \gamma_t, s_{t+1}), \quad \gamma_t \geq 0, t \geq 0 \quad (3)$$

$$\mu_0 = 0, \qquad x_0, s_0 \text{ given},$$

where the mappings h and φ are derived from r, g and N_j.

- *Recursive* since no future variables in constraints, and all functions in constraints are known.

Saddle Point Functional Equation (SPFE)

- Under certain assumptions solutions to Program 2 obey an SPFE $\Rightarrow \exists! W(x, \mu, s)$ satisfying

$$\begin{aligned} W(x,\mu,s) &= \inf_{\gamma \geq 0} \sup_{a \in A(x,s)} \left\{ h(x,a,\mu,\gamma,s) + \beta \mathbb{E}[W(x',\mu',s')|s] \right\} \\ \text{s.t.} \quad & x' = \ell(x,a,s) \\ \text{and} \quad & \mu' = \varphi(\mu,\gamma,s') \end{aligned}$$

  $\forall (x, \mu, s)$ such that $W(x_0, \mu_0, s_0)$ is the value of Program 2 for initial conditions (x_0, μ_0, s_0).
- $\Rightarrow$ Policy correspondence: $(a_t, \gamma_t) = \psi(x_t, \mu_t, s_t)$.

Partnership with limited commitment

- Consider a model where J agents share risks and jointly invest in a project which cannot be undertaken unless all agents participate
- Agent j receives endowment $\omega_{j,t}$ at time t
- Total production: $F(k, \theta)$, where θ is a productivity shock
- Joint process $\{\theta_t, \omega_t\}_{t=0}^{\infty}$ Markovian
- Initial conditions $(k_0, \theta_0, \omega_0)$ given

Write as Program 1

Program 1

- $r(x, a, s) \equiv \sum_{j \in J} \alpha_j u(c_j)$, where α is the Pareto weight
- $s \equiv (\theta, \omega)$ (exogenous state)
- $x \equiv k$ (endogenous state)
- $\ell(x, a, s) \equiv (1 - \delta)k + i$ (evolution eqn.)

Write as Program 1 (cont'd)

Program 1

- $a \equiv (i, c)$ (decision var.)

$$A(x,s) \equiv \left\{ (i,c) \geq 0 : \sum_{j \in J} c_j + i \leq F(k,\theta) + \sum_{j \in J} \omega_j \right\}$$

- Participation constraint, to map into (2):

$$\mathbb{E}_t \sum_{n=0}^{\infty} \beta^n u(c_{j,t+n}) \geq v_j^a(\omega_t) \qquad \text{for all } \; j, t$$

$$\Rightarrow g_1^j(x,a,s) \equiv u(c_j) - v_j^a(\omega) \text{ and } g_2^j(x,a,s) \equiv u(c_j)$$

Write as Program 2

▸ Program 2

Using previous algebra:

$$L \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_t \sum_J [(\alpha_j + \mu_{j,t}) u_j(c_{j,t}) + \gamma_{j,t}(u(c_{j,t}) - v_j^a(\omega_t))]$$

So we have

$$\begin{aligned} h(x, a, \mu, \gamma, s) &\equiv \sum_{j \in J} ((\alpha_j + \mu_j + \gamma_j) u(c_j) - \gamma_j v_j^a(\omega)) \\ \varphi(\mu, \gamma, s) &\equiv \mu + \gamma \quad (= \mu') \end{aligned}$$

Solution from SPFE

$$W(k,\mu,\omega,\theta) = \inf_{\gamma\geq 0} \sup_{c,i} \left\{ \sum_{j\in J} ((\alpha_j+\mu_j+\gamma_j)u(c_j) - \gamma_j v_j^a(\omega)) + \beta\mathbb{E}[W(k',\mu',\omega',\theta')|\omega,\theta] \right\}$$

$$\text{s.t.} \quad k' = (1-\delta)k + i$$

$$\sum_{j\in J} c_j + i \leq F(k,\theta) + \sum_{j\in J} \omega_j$$

$$\mu' = \mu + \gamma$$

With initial conditions $(k_0, \mu_0 = 0, \theta_0, \omega_0)$.

Solution

- With full enforcement:

$$\frac{u'(c_{i,t})}{u'(c_{j,t})} = \frac{\alpha_j}{\alpha_i}$$

- With participations constraints:

$$\frac{u'(c_{i,t})}{u'(c_{j,t})} = \frac{\alpha_j + \mu_{j,t+1}}{\alpha_i + \mu_{i,t+1}}$$

- Co-state variable μ tracks binding PC's and wealth distribution