Stationary Equilibria in Asset-Pricing Models with Incomplete Markets and Collateral

Kubler and Schmedders - Econometrica 2003

Sargent's Reading Group

NYU - 17 April 2007

Motivation

- Asset pricing model with:
  - infinitely lived heterogeneous agents;
  - collateral constraints and default;
  - incomplete markets.

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- Algorithm to approximate this mapping numerically.
- Example of a simple economy; welfare gains from collateralization and equilibrium default.

Model

- $1, \ldots, H$ agents. Each h maximizes $E\left\{\sum_{t=0}^{\infty} \beta_h^t u_h(c_t, y_t)\right\}$ with $u_h(\cdot, y)$ unbounded below $\forall y$.

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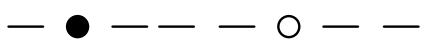
- $1, \ldots, J$ financial assets. If h sells 1 unit of security j at node σ, he must hold $k_a^j(\sigma)$ dollars worth of each tree a, with k_a^j a stationary function of current shock y and current endogenous variables.

Model



y realized:

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$$f_j(\sigma) = \min \left\{ b_j(y), \sum_a k_a^j(\sigma^*) \frac{q_a(\sigma)}{q_a(\sigma^*)} \right\},$$ i.e. partial default

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Financial Markets Equilibrium

- For all σ, h, a, j, a collection of:
  - share holdings $\theta_a^h(\sigma) \geq 0$ and asset holdings $\phi_j^h(\sigma)$;
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- Each agent h maximizes expected utility subject to:

$$i) \quad c^h(\sigma) + \sum_a \theta_a^h(\sigma) \left[q_a(\sigma) - d_a^h(y) \right] + \sum_j \phi_j^h(\sigma) p_j(\sigma) =$$

$$= e^h(y) + \sum_j \phi_j^h(\sigma^*) f_j(\sigma) + \sum_a \theta_a^h(\sigma^*) q_a(\sigma), \quad \forall \sigma$$

$$ii) \quad q_a \theta_a^h(\sigma) + \sum_{j: \phi_j^h(\sigma) < 0} k_a^j \phi_j^h(\sigma) \geq 0, \quad \forall a$$

Existence of Markov Equilibrium

- Financial wealth with $\Omega(\sigma) = \big(\omega^1(\sigma), \ldots, \omega^H(\sigma)\big)$:

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- Markov equilibrium:
  - nonempty valued policy correspondence mapping from current exogenous shock y and financial wealth distribution $\Omega(\sigma)$ to current prices $q_a(\sigma), p_j(\sigma)$ and choices $c^h(\sigma), \theta_a^h(\sigma), \phi_j^h(\sigma)$;
  - transition function mapping from current state to tomorrow's endogenous states, consistent with optimality;
  - "simple" if the associated policy correspondence is single valued.

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- Markov equilibrium:
  - Lemma: is a financial markets equilibrium;
  - Theorem: there exists a Markov equilibrium with nonempty policy correspondence.

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- Practical improvement over existing algorithms, which cannot consider models with more than two assets. Here, dimension of the domain of the equilibrium map is independent of the of the number of securities traded.

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  - compute the new approximation ρ^{n+1} by interpolation;
  - if $sup_{y,\Omega} \parallel \rho^{n+1} - \rho^n \parallel < \delta$ not satisfied, set $n = n + 1$; if it is satisfied, compute the maximum relative error in the Euler equations; if it is not below ε, start again with lower δ; if it is, stop.

Example

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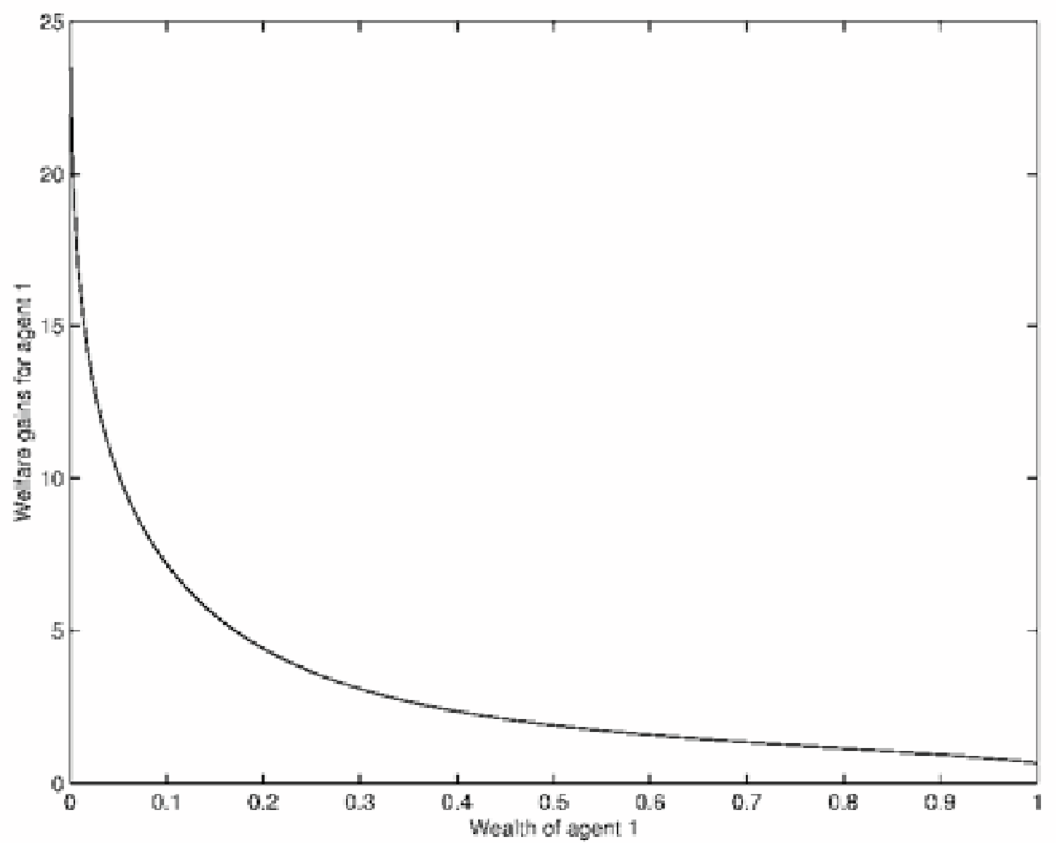
- Endowments $e^1 = (1.386; 2.205; 5.544; 5.145)$; $e^2 = 0.7\bar{e} - e^1$.
- Transition probabilities:

$$\begin{pmatrix} .4 & .4 & .1 & .1 \\ .4 & .4 & .1 & .1 \\ .1 & .1 & .4 & .4 \\ .1 & .1 & .4 & .4 \end{pmatrix}$$

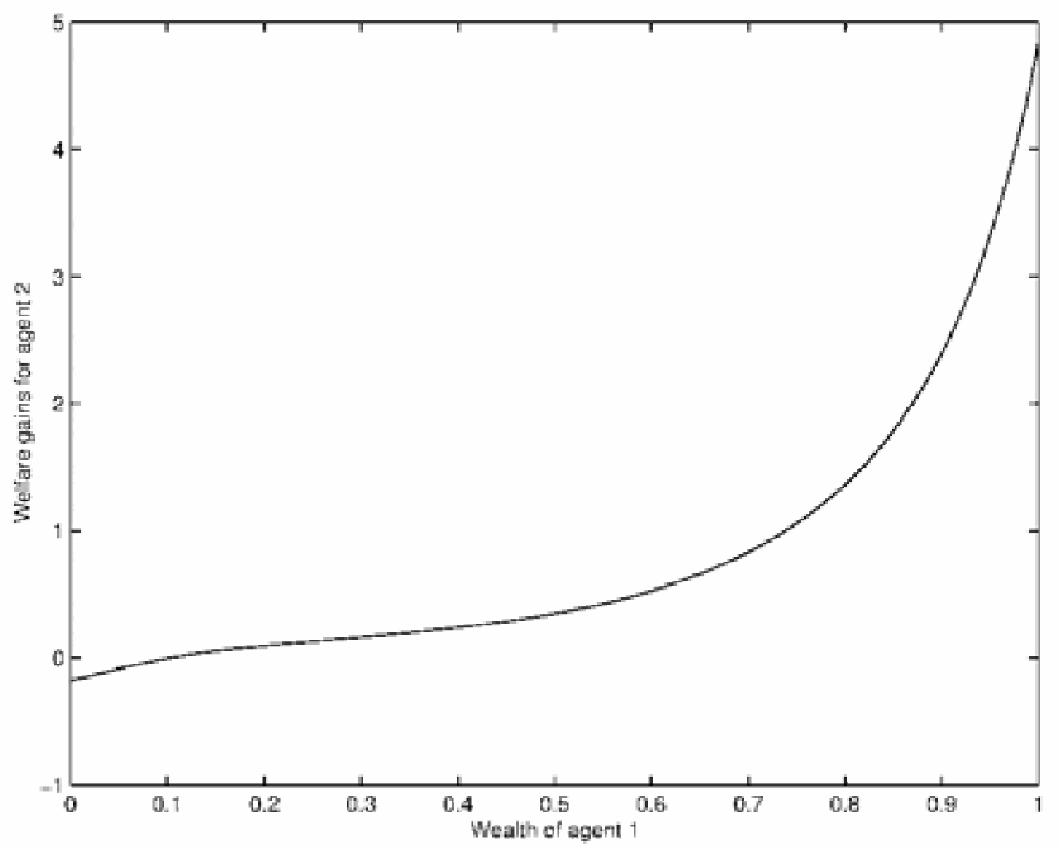
Example: welfare gains

- Welfare gains (for agent 1) from introducing the possibility of trading 1 risk-free bond backed by collateral ($k = 1.3$ across states) and with $y_0 = 1$.



Example: welfare gains

- Welfare gains (for agent 2) from introducing the possibility of trading 1 risk-free bond backed by collateral ($k = 1.3$ across states) and with $y_0 = 1$.



Example: margin requirements

- Default rates as a function of k.
- Welfare gains and losses (in %) with respect to $k = 1.3$ and for different initial wealth levels.

k	1.02	1.05	1.1	1.15	1.2
Default rate	25.9	19.2	5.9	0.05	0
$\Omega = (0.1, 0.9)$	$(-0.12, 0.08)$	$(-0.07, 0.05)$	$(0.11, 0.03)$	$(0.32, 0.03)$	$(0.19, 0.02)$
$\Omega = (0.5, 0.5)$	$(-0.01, 0.0)$	$(-0.01, 0.0)$	$(0.0, 0.0)$	$(0.02, 0.01)$	$(0.01, 0.0)$
$\Omega = (0.9, 0.1)$	$(0.05, -0.05)$	$(0.03, -0.01)$	$(0.02, 0.02)$	$(0.02, 0.03)$	$(0.01, 0.02)$

Example: some remarks

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- Portfolio policy functions (300 knots for the spline approx.):