Can Interest Rate Volatility be Extracted from the Cross Section of Bond Yields?

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What is the paper about?

- Investigates the relation between interest rate volatility and the cross-section of bond yields.
- Two facts:
  - Three factors are needed to capture variation in bond yields.
  - Interest rate volatility is stochastic.
- In the standard affine models state variables that drive changes in interest rate volatility play a ‘dual role’ in that they also drive changes in bond yields.
- Questions?
  - Can the variance state variable in standard affine models satisfy its dual role?
  - Investigate a subset of the affine class that displays ‘unspanned stochastic volatility’.

Overview

- Affine Term Structure Models
- Unspanned Stochastic Volatility
- Data
- Estimation Algorithm
- Some Results

Affine Term Structure Models

The time-t price of a zero-coupon bond that matures in τ periods.

$$P(t,\tau) = E_t^Q \left[e^{-\int_t^\tau r_s ds} \right]$$

The instantaneous short rate is an affine function of a vector of N state variables X_t.

$$r_t = \delta_0 + \delta_x' X_t$$

X_t follows an "affine diffusion"

$$dX_t = \mathcal{K}^Q (\Theta^Q - X_t) dt + \Sigma \sqrt{S_t} dZ_t^Q$$

S is a diagonal matrix with the ith diagonal element given by

$$[S_t]_{ii} = \alpha_i + \beta_i' X_t$$

Affine Term Structure Models

Zero coupon bond prices

$$P(t,\tau) = e^{A(\tau)-B(\tau)' X_t}$$

where $A(\tau)$ and $B(\tau)$ satisfy the ODE's:

$$\begin{aligned}
\frac{dA(\tau)}{d\tau} &= -\Theta^{Q\top}\mathcal{K}^{Q\top} + \frac{1}{2}\sum_{i=1}^{N}\left[\Sigma^{\top}B(\tau)\right]_i^2 \alpha_i - \delta_0 \\
\frac{dB(\tau)}{d\tau} &= -\mathcal{K}^{Q\top} - \frac{1}{2}\sum_{i=1}^{N}\left[\Sigma^{\top}B(\tau)\right]_i^2 \beta_i + \delta_x
\end{aligned}$$

and initial conditions:

$$A(0) = 0, \qquad B(0) = 0$$

Prices of risk in the model

The state price deflator π_t has dynamics given by,

$$\frac{d\pi_t}{dt} = -r_t dt - \Lambda_t' dZ_t^P$$

Z_t^P follows a Brownian motion under the physical measure.

The dynamics of the state vector under P are.

$$dX_t = ((\mathcal{K}\Theta)^Q - \mathcal{K}^Q X_t))dt + \Sigma\sqrt{S_t}\Lambda_t dt + \Sigma\sqrt{S_t} dZ_t^P$$

Instantaneous bond price dynamics can be written as

$$\frac{dP(X_t,\tau)}{P(X_t,\tau)} = (r_t + e_{\tau,t})dt + \nu_{\tau,t} dZ_t^P$$

where

$$e_{\tau,t} = -B(\tau)'\Sigma S_t \Lambda_t$$

$A_m(N)$ admissible models of Dai and Singleton

- The parameter vector $\Phi \equiv (\mathcal{K}, \Theta, \Sigma, \beta, \alpha)$ might not be admissible.
- $[S(t)]_{ii}$ may not be positive over the range of X.
- Admissibility is an issue only when $\beta_i \neq 0$.
- Impose constrains on the drift parameters ($\mathcal{K}$ and Θ) and diffusion coefficients (Σ and β).
- Let $m = \text{rank}(\beta)$. Dai and Singleton denote as $A_m(N)$ the set of admissible models with N factors.

An admissible system is still subject to invariant affine transformations.

State Vector

Choose a representation for the state vector in which each state variable has a clear economic interpretation.

$$X_t = [r_t, \mu_t^Q, \theta_t^Q, V_t]'$$

where,

$$\begin{aligned} r_t &= Y_t(0) \\ \mu_t^Q &= 2\frac{\partial Y_t(\tau)}{\partial \tau}\bigg|_{\tau=0} \\ \theta_t^Q &= 3\frac{\partial^2 Y_t(\tau)}{\partial \tau^2}\bigg|_{\tau=0} \\ V_t &= \left(\frac{1}{dt}\right) dr_t^2 \end{aligned}$$

The $A_1(3)$ model

State vector is: $X_t = [r_t, \mu_t^Q, V_t]^\top$

$$\frac{1}{dt} E^Q[dX_t] = \begin{bmatrix} \mu_t^Q \\ m_0 + m_r r_t + m_\mu \mu_t^Q + m_v V_t \\ \gamma_V - \kappa_V V_t \end{bmatrix}$$

and

$$\frac{1}{dt} \text{Cov}\left(dX_t, dX_t^\top\right) \equiv \Omega_t = \Omega_0 + \Omega_V (V_t - \underline{V})$$

$\underline{V}$ is the lower bound of the V_t process and,

$$\Omega_0 = \begin{bmatrix} \underline{V} & \underline{c}_{r\mu} & 0 \\ \underline{c}_{r\mu} & \underline{\sigma}_u & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Omega_V = \begin{bmatrix} 1 & c_{r\mu} & c_{rV} \\ c_{r\mu} & \sigma_\mu & c_{\mu V} \\ c_{rV} & c_{\mu V} & \sigma_V \end{bmatrix}$$

The $A_1(4)$ model

State vector is: $X_t = [r_t, \mu_t^Q, \theta_t^Q, V_t]^\top$

$$\frac{1}{dt} E^Q[dX_t] = \begin{bmatrix} \mu_t^Q \\ \theta_t^Q + V_t \\ a_0 + a_r r_t + a_\mu \mu_t^Q + a_\theta \theta_t^Q + a_v V_t \\ \gamma_V - \kappa_V V_t \end{bmatrix}$$

and

$$\frac{1}{dt} \mathrm{Cov}\left(dX_t, dX_t^\top\right) \equiv \Omega_t = \Omega_0 + \Omega_V (V_t - \underline{V})$$

$\underline{V}$ is the lower bound of the V_t process and,

$$\Omega_0 = \begin{bmatrix} \underline{V} & \underline{c}_{r\mu} & \underline{c}_{r\theta} & 0 \\ \underline{c}_{r\mu} & \underline{\sigma}_u & \underline{c}_{\mu\theta} & 0 \\ \underline{c}_{r\theta} & \underline{c}_{\mu\theta} & \underline{\sigma}_\theta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Omega_V = \begin{bmatrix} 1 & c_{r\mu} & c_{r\theta} & c_{rV} \\ c_{r\mu} & \sigma_\mu & c_{\mu\theta} & c_{\mu V} \\ c_{r\theta} & c_{\mu\theta} & \sigma_\theta & c_{\theta V} \\ c_{rV} & c_{\mu V} & c_{\theta V} & \sigma_V \end{bmatrix}$$

The $A_1(4)$ Unspanned stochastic volatility (USV) model

Motivated by the fact that the factors that explain variation in yields account for only a small part of the variation in interest rate options.

Imposes very tight parametric restrictions

$$
\begin{aligned}
a_r &= -2c_{r\mu}^2(3c_{r\mu} - a_\theta) \\
a_\mu &= 7_{r\mu}^2 - 3c_{r\mu}a_\theta \\
a_V &= 3c_{r\mu} \\
\sigma_\mu &= c_{r\mu}^2 \\
\sigma_\theta &= c_{r\mu}^4 \\
c_{r\theta} &= cr\mu^2 \\
c_{\mu\theta} &= c_{r\mu}^3
\end{aligned}
$$

Proposition

$$P(t,\tau) = \exp\left(A(\tau) - B_r(\tau)r_t - B_\mu(\tau)\mu_t^Q - B_\theta(\tau)\theta_t^Q\right)$$

Data

- Weekly LIBOR and swap rate data from 1998 to 2005.
- Zero coupon yields are first bootstrapped.
- Models are estimated using data from 1988 to 2002.
- Data from 2003 to 2005 is saved for out-of sample analysis.

A Metropolis within Gibbs MCMC algorithm

- Given a panel of zero coupon yields $Y^T = [Y_1, Y_2, ..., Y_T]$ the goal is to compute
$$p(\phi|Y) \propto p(Y|\phi)p(\phi)$$
- Augment the observable data with the term structure factor data $X^T = [X_1, X_2, ..., X_T]$.
- Integrate out the $X's$ using a Gibbs sampler that alternates between performing draws from $p(\phi|Y^T, X^T)$ and $p(X^T|Y^T, \phi)$.
- Further break up the parameter vector into three components ϕ^Q,ϕ^λ,andϕ^Λ.
- Decompose the state vector as:
$$X_t = \begin{bmatrix} X_t^0 \\ V_t \end{bmatrix}$$
- V_t, ϕ^Q are drawn using a MH algorithm.

In and out-of-sample yield fit

	$A_1(3)$		$A_1(4)$ USV		$A_1(4)$
			In-sample RMSE (basis points)		
6-month	3.39	$<^{**}$	4.85	$>^{**}$	0.94
1-year	4.48	$<^{**}$	6.05	$>^{**}$	1.94
2-year	2.52	$<^{**}$	3.81	$>^{**}$	1.13
3-year	1.42	$<^{*}$	1.63	$>^{**}$	1.08
4-year	2.56	$<^{**}$	3.86	$>^{**}$	0.54
5-year	2.86	$<^{**}$	4.34	$>^{**}$	0.95
7-year	1.57	$<^{**}$	1.97	$>^{**}$	1.31
10-year	3.95	$<^{**}$	5.92	$>^{**}$	1.23
			Out-of-sample RMSE (basis points)		
6-month	5.13	$>^{**}$	2.99	$>^{**}$	0.52
1-year	5.77	$>^{*}$	3.98	$>^{**}$	0.86
2-year	4.98	$>^{**}$	1.90	$>^{*}$	0.69
3-year	1.12		1.27	$>^{**}$	0.39
4-year	4.61	$>^{**}$	2.23	$>^{**}$	0.48
5-year	5.44	$>^{**}$	2.28	$>^{**}$	0.57
7-year	1.97	$>^{*}$	1.36	$>^{**}$	0.49
10-year	7.09	$>^{**}$	3.20	$>^{**}$	0.65

Correlations of observed and model implied time series

Table 4: Correlations of observed and model-implied time series

This table reports correlations between actual and model-implied series computed over the 1988 to 2002 sample period. Average yield is simply the average of the .5, 1, 2, 3, 4, 5, 7, and 10-year zero yields. Slope is defined as the 10-year yield minus the 6-month yield. Curvature is defined using the 3-year yield in addition. Rolling 30-day window and GARCH(1,1) volatilities are calculated from demeaned daily changes in the six-month rate. The implied volatility series is obtained from short-term Eurodollar futures options. These data are only available starting in 1991, so correlations involving implied volatilities are calculated over the 1991-2002 sample period.

	$A_1(3)$	$A_1(4)$ USV	$A_1(4)$	Daily GARCH	Eurodollar implied vol
Actual vs. model average yield	1.000	1.000	1.000		
Actual vs. model slope	0.998	0.998	1.000		
Actual vs. model curvature	0.998	0.997	1.000		
Rolling vs. model volatility	-0.600	0.783	0.156	0.957	0.676
GARCH vs. model volatility	-0.587	0.786	0.140	1.000	0.693
Eurodollar implied vs. model volatility	-0.498	0.605	0.377	0.693	1.000
Actual curvature vs. model volatility	0.275	-0.087	-0.065	-0.052	-0.103
Actual curvature vs. model variance	0.285	-0.072	-0.073	-0.020	-0.124

out-of-sample volatility forecasts

	$A_1(3)$		$A_1(4)$ USV		$A_1(4)$
			Out-of-sample RMSE of weekly $\lvert\Delta Y\rvert$		
6-month	7.17	$>^{**}$	4.23	$<^{**}$	5.87
1-year	7.11	$>^{**}$	5.22	$<^{**}$	6.37
2-year	8.00	$>^{**}$	7.18	$<^{*}$	7.40
3-year	8.45	$>^{**}$	8.07		8.00
4-year	8.71	$>^{*}$	8.48	$>^{**}$	8.37
5-year	8.86		8.67	$>^{*}$	8.57
7-year	8.88		8.67		8.62
10-year	8.71		8.42		8.42
			Out-of-sample RMSE of $\hat{\sigma}$		
6-month	8.66	$>^{**}$	4.69	$<^{**}$	6.95
1-year	7.40	$>^{**}$	4.54	$<^{**}$	6.35
2-year	7.01	$>^{**}$	5.64	$<^{**}$	6.00
3-year	6.97	$>^{**}$	6.27		6.18
4-year	7.00	$>^{**}$	6.53	$>^{**}$	6.37
5-year	7.02	$>^{**}$	6.62	$>^{*}$	6.48
7-year	6.92	$>^{*}$	6.50		6.42
10-year	6.60	$>^{*}$	6.08		6.05

Model versus actual volatility