If You're So Smart, Why Aren't You Rich ? Belief Selection in Complete and Incomplete Markets

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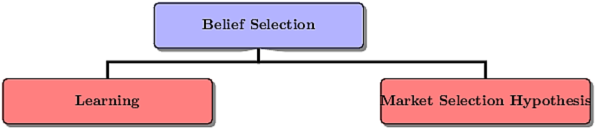
Motivation

Two basic hypothesis for economic theory

1. Expected Utility Hypothesis - Agents maximize **expected utility** with respect to some beliefs
2. Rational Expectations Hypothesis - Beliefs of all the agents are **correct**

What justifies correct beliefs ?



Market Selection Hypothesis

1. Do markets favor rational traders ?
   - DeLong, Shleifer, Summers and Waldman (1990,91)
   - Sandroni (2000)
2. What are the necessary and sufficient conditions for *survival* ?
   - Heterogeneity in Beliefs?
   - Market Completeness ?
   - Heterogeneity in Preferences - Attitudes towards time/risk?
   - Learning Rules ?

Insight : *Understand Market Selection through Pareto Optimality*

Notation

1. Time : $\mathcal{T} = \mathbb{Z}_+$
2. States : $\mathcal{S} = \{1, \dots, S\}$
3. Probability Space : $\langle \Sigma, \mathcal{B}(\Sigma), p \rangle$

$$\Sigma = \{\sigma | \sigma = (\sigma_0, \dots) \in \mathcal{S}^\infty\}$$

$$\sigma^t = (\sigma_0, \dots, \sigma_t)$$

4. Consumption Plan c : $\Sigma \to \mathbb{R}_+^\infty$ s.t

$$c_t(\sigma) \text{ is } \quad \mathcal{F}_t \text{ measurable}$$

5. Agents : $\mathcal{I} = 1, \dots, I$ where each agent is described by $\{e^i, \beta_i, u^i, p^i\}$

$$U^i(c) = \mathbb{E}_{pi} \left\{ \sum_{t=0}^{\infty} \beta_i^t u^i [c_t(\sigma)] \right\}$$

Concepts and Definitions

- **Absolute continuity of measures** : p is absolutely continuous w.r.t q if

$$q(A) = 0 \implies p(A) = 0 \quad \forall A \in \mathbb{B}(\Sigma)$$

- **Relative Entropy of measures** : The relative entropy of q w.r.t p

$$I(p, q) = \mathbb{E}_q \left[\frac{dp}{dq} \log \frac{dp}{dq} \right]$$

where $\frac{dp}{dq}$ is the Radon-Nikodym derivative for p wrt q

- **Survival**: Agent i survives on path σ iff $\limsup_t c_t^i(\sigma) > 0$
- **Vanishing** Agent i vanishes on path σ iff $\lim_t c_t^i(\sigma) = 0$
- **Rational Expectations** Agent i has rational expectations if $p^i = p$

Assumptions

- **A.1.** Restrictions on preferences : u^i strictly concave,strictly monotonic and satisfy an Inada condition at 0
- **A.2.** Uniform bounds on the aggregate endowment

$$\infty > F = \sup_{t,\sigma} \sum_i e^i(\sigma) \geq \inf_{t,\sigma} \sum_i e^i_t(\sigma) = f \geq 0$$

- **A.3.** Absolute continuity over finite horizons $\forall \quad i, t, \sigma$

$$p_t(\sigma) > 0 \implies p^i_t(\sigma) > 0$$

Planner's Problem Problem

$$\max_{c^1 \dots c^I} \sum \lambda_i U^i(c)_i$$

s.t

$$\sum_i c^i - e \leq 0$$

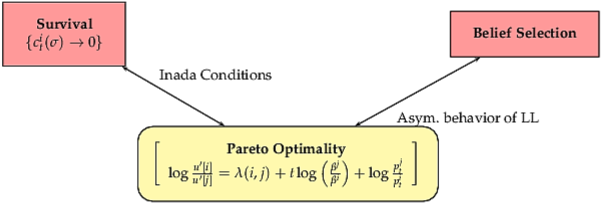
$$c_t^i(\sigma) \geq 0$$

FOC : $\forall t, \sigma$

$$\lambda_i \beta_i^t u^{i'}[c_t^i(\sigma)] p_t^i(\sigma) - \mu_t(\sigma) = 0$$

$$p_t^i(\sigma) = 0 \implies c_t^i(\sigma) = 0$$

Survival, Optimality and Beliefs



Survival and MRS

Characterize the necessary and sufficient conditions for vanishing events : $\lim_t c_t^i(\sigma) = 0$

- Sufficient :

$$\lim_t \frac{u^{i'}[c_t^i(\sigma)]}{u^{j'}[c_t^j(\sigma)]} = \infty$$

- Necessary : for some j,

$$\limsup_t \frac{u^{i'}[c_t^i(\sigma)]}{u^{j'}[c_t^j(\sigma)]} = \infty$$

Simple Example - IID beliefs

With IID beliefs $p_t(\sigma) = \prod_{\tau=0}^{t} \rho(\sigma_t)$ ...
Pareto Optimality $\implies$

$$\frac{1}{t}\log\frac{u^{i\prime}[c_t^i(\sigma)]}{u^{j\prime}[c_t^j(\sigma)]} = \frac{1}{t}\lambda(i,j) + \log\frac{\beta_j}{\beta_i} + \frac{1}{t}\log\frac{p_t^j}{p_t^i}$$

$$\frac{1}{t}\log\frac{u^{i\prime}[c_t^i(\sigma)]}{u^{j\prime}[c_t^j(\sigma)]} = \underbrace{\frac{1}{t}\lambda(i,j)}_{\mathbf{=0}} + \underbrace{\left(\log\beta_j + \frac{1}{t}\log p_t^j\right)}_{\kappa_j} - \underbrace{\left(\log\beta_i + \frac{1}{t}\log p_t^i\right)}_{\kappa_i}$$

Note that

$$\frac{1}{t}\log p_t^i = \frac{1}{t}\sum_{\tau=0}^{t}\log\rho^i(\sigma_t) \underbrace{\rightarrow}_{LLN} \mathbb{E}_\rho \log\rho^i = \mathbb{E}_\rho\log(\rho) - I_\rho(\rho^i)$$

Simple Example - Lessons

We can interpret $\kappa_i = \beta_i - I_\rho(\rho^i)$ as *survival index*
For large t,

$$\frac{u^{i\prime}[c_t^i(\sigma)]}{u^{j\prime}[c_t^j(\sigma)]} \approx e^{(\kappa_j - \kappa_i)t}$$

1. Necessary Condition for survival : $\kappa_i = \max\{\kappa_j\}$
2. κ^i is increasing in β_i : Patient agents live longer
3. κ^i is decreasing in $I_\rho(\rho^i)$: Survival chances rise with *closeness* to truth
4. Attitudes towards risk do not matter for survival, they only affect the rate of survival

Rational Expectations and Survival

Let $\beta_i = \beta_j$

What are the necessary and sufficient conditions on beliefs for survival ?

$\longrightarrow$ **Absolute Continuity** with respect to truth

1. Sufficiency : $\frac{p_t^j}{p_t^i}$ is a p^i- positive (integrable) martingale and hence converges a.s
2. Necessary (In presence of a RE agent) : If not then $\exists A$ s.t $p^i(A) = 0$ and $p^{RE}(A) > 0$
   This makes the RHS unbounded

Learning consistency and Absolute Continuity

Absolute continuity is necessary only in presence of a RE Agent. Is this restrictive ?

For e.g.: Consider a Bayesian Agent - $\{\Theta^i, \mu^i, (p^\theta)_{\theta \in \Theta^i}\}$

$$p_t^i(\sigma) = \int_{\Theta^i} p_t^\theta(\sigma) d\mu^i(\theta)$$

If $\Theta^i \subset \mathcal{R}^d$ and μ^i is abs. cts wrt to Lebesgue measure, we may have that p^i is *mutually singular* w.r.t to any particular p^θ.

For e.g. : Consider a infinite sequence of IID coin toss X_i with θ as the probability of winning

$$p^i \left\{ \lim \frac{1}{t} \sum X_i = \theta \right\} = 0$$

but

$$p^\theta \left\{ \lim \frac{1}{t} \sum X_i = \theta \right\} = 1$$

Survival with Learning

What are the necessary and sufficient conditions for survival with learning ?

1. **Sufficiency** : A Bayesian survives for μ^i almost θ

   Does this contradict the earlier analysis with a RE agent ?

2. **Necessary** : In presence of a Bayesian agent with prior μ^B, if Agent i survives for μ^B- almost θ, then p^B is abs cts w.r.t p^i

   Being a Bayesian is not necessary for survival but forecasts should merge with a Bayesian eventually

Example of a Non Bayesian Survivor

Consider an Binary IID environment ($X_t(\sigma) \in \{1, 2\}$) with $p^\theta\{X_t(\sigma) = 2\} = \theta$.

$$p^i_t(\sigma) = p^{m_t(\sigma)}$$

$$m_t(\sigma) = \frac{1}{T+1} \sum_{t=0}^{T} \left(\frac{X_t(\sigma) - 1}{T+1} \right)$$

Agent i here is using MLE estimates to form beliefs but is not a Bayesian

- Consider two paths - $\{\sigma^T : X_t(\sigma) = 1 \quad \forall\ t \leq T\}$ and $\{\tilde{\sigma}^T : X_t(\tilde{\sigma}) = 2 \quad \forall t \leq T\}$

$$p^i_T(\sigma) = p^0 \quad \& \quad p^i_T(\tilde{\sigma}) = p^1$$

Under the IID hypothesis, Agent i cannot be a Bayesian but his forecasts merge with a Bayesian (say with a Dirichlet prior)

Market Selection amongst Bayesian

Does the Market selection work similarly for all Bayesian ?

- Bayesian with abs cts priors share the same fate
- However Bayesian with larger model spaces are at disadvantage

Proposition

Suppose Agent K has a prior belief abs. cts w.r.t the Lebesgue measure on Θ^K and Agent k has a similar representation on Θ^k. If $dim(\Theta^k) < dim(\Theta^K)$, then

- μ^K *a.s* $\quad \theta \in \Theta^K \cap [\Theta^k]^c$ *Agent k vanishes*
- $\forall \theta \in \Theta^k$ *Agent K vanishes in probability*

Rates of Learning-IID case

Rewrite the Log Likelihood Ratio

$$\log \frac{p_t^j}{p_t^i} = \log \frac{p_t^\theta}{p_t^i} - \log \frac{p_t^\theta}{p_t^j}$$

$$\frac{1}{t} \log \frac{p_t^\theta}{p_t^i} \to I_{p^\theta}(p^i)$$

So $\frac{p_t^\theta}{p_t^i}$ diverges p^θ a.s. [SLLN] Now is there a $f(t)$ such that

$$\frac{1}{f(t)} \frac{p_t^\theta}{p_t^i}$$

converges to something non-degenerate ?

Proposition

Clarke and Barron Theorem :

$$\log \frac{p_t^\theta}{p_t^i} - \left(\frac{d}{2} \log \frac{t}{2\pi} + \frac{1}{2} \log \|\mathcal{I}(\theta)\| - \log \mu(\theta) \right) \underset{p}{\to} \chi^2(d)$$

Heterogeneous Discount factors

Bad forecasts are offset by higher discount factors. So with Heterogeneous discount factors relative rates matter

- Discount factors - $t \log \frac{\beta_j}{\beta_i} \approx$ *Linear in t*
- Log Likelihoods - $\log \frac{p_t^j}{p_t^i} \approx$ *Linear in* $\log t$

Discount factor differences dominate asymptotically

Incomplete Markets

1. No longer appeal to Pareto Optimality - Survival has to be checked directly in a decentralized setups
2. Excessive Savings may help survival of irrational agents. Further it may drive out rational agents too !
3. Disconnect between growth rate of optimal portfolios and relative entropy of beliefs

Extension 1 : Dominant root and rates of learning

Pareto Optimality allows us to focus on the behavior of

$$\underbrace{\left(\frac{\beta_j}{\beta_i}\right)^t}_{M_{0,t}} \underbrace{\left(\frac{p_t^j(\sigma)}{p_t(\sigma)}\right)}_{M_{j,t}} \underbrace{\left(\frac{p_t(\sigma)}{p_t^i(\sigma)}\right)}_{M_{i,t}^{-1}}$$

Observations : For an underlying Markovian environment,

- $M_{i,t}$ are multiplicative process i.e

$$M_{i,t+s} = M_{i,t} \cdot \mathbb{L}^{+t}(M_i)_s$$

- $M_t = M_{0,t} M_{i,t} M_{j,t}^{-1}$ is also multiplicative
- Further for a diffusion setup, A.3 and Girsanov theorem give us a closed form for M

Extension 1 : Dominant root and rates of learning

$$dM_{0,t} = -M_0(\delta_j - \delta_i)dt$$
$$dM_{i,t} = -M_{i,t}\xi_t^i dB_t$$

Hansen Schienkman (2009) illustrate a multiplicative decomposition into the following components

$$M_t = e^{\lambda t}\tilde{M}_t \frac{e(X_0)}{e(X_t)}$$

$$\frac{1}{t}\log \mathbb{E} M_t = \lambda + \frac{1}{t}\log \hat{\mathbb{E}}_t e^{-1}(X) + o(t^{-1})$$

Under some regularity conditions(S.S under hat measure), we have the (geometric) rate at which EM_t diverges. Now if we can show that M_t is uniformly integrable, we also have a.s divergence.
λ *here acts as the relative survival index*

Extension 2 : Ambiguity and Belief Selection

Knightian uncertainty refers to environments where agents have multiple beliefs.

- Not a clear demarcation between survival events and asym. divergence of LL ratios.
- Risk Perceptions depend on Risk sharing and the optimal allocation has to respect this link
- In some ways agent's are endogenously doing a belief selection looking at their continuation values and market is doing a belief selection via wealth transfers. *The interaction between these process is non-trivial*

Concepts and Definitions - EXTRA

- Absolute continuity over infinite horizon has strong implications for mutual agreement of conditional distributions over time. Blackwell Dubins(1962) refer to this as the *merging* of two distributions.
- Relative entropy acts like a distance measure for distributions but is not a metric. It however is non-negative, convex and singular iff the arguments are same.