The Combinatorial Assignment Problem

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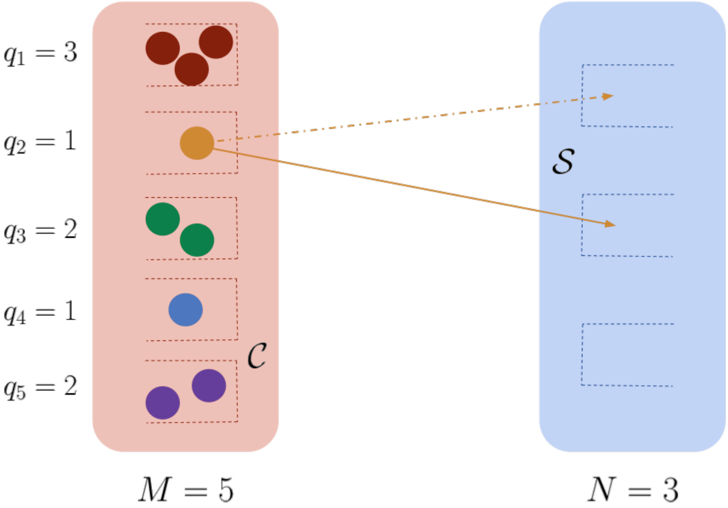
Terminology

$$\mathcal{E} = \left(\mathcal{S}, \mathcal{C}, (q_j)_{j=1}^{M}, (\Psi_i)_{i=1}^{N}, (u_i)_{i=1}^{N}\right)$$



Wish List

Equilibrium Concept:

[ ] Existence

[ ] Efficiency

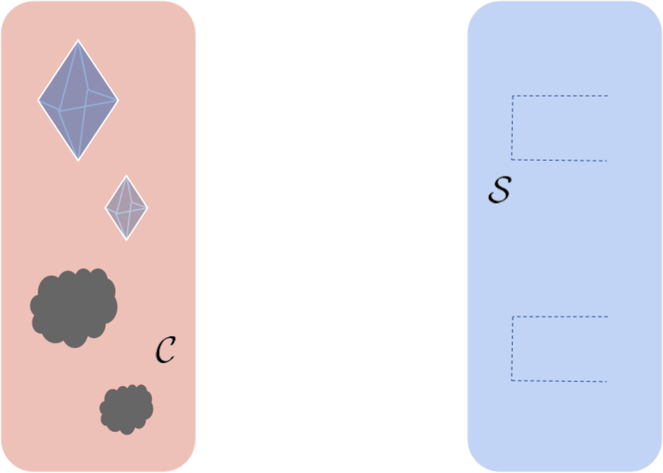
[ ] Fairness

Mechanism:

[ ] Strategy Proof-ness

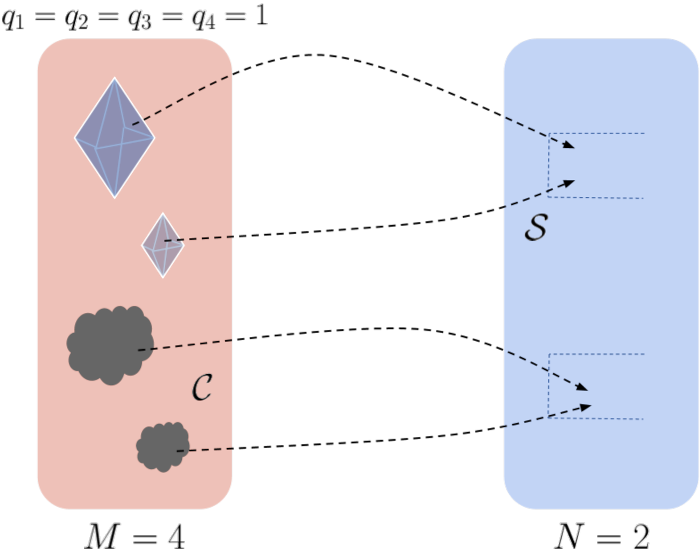
Example (Indivisible Goods)

$q_1 = q_2 = q_3 = q_4 = 1$

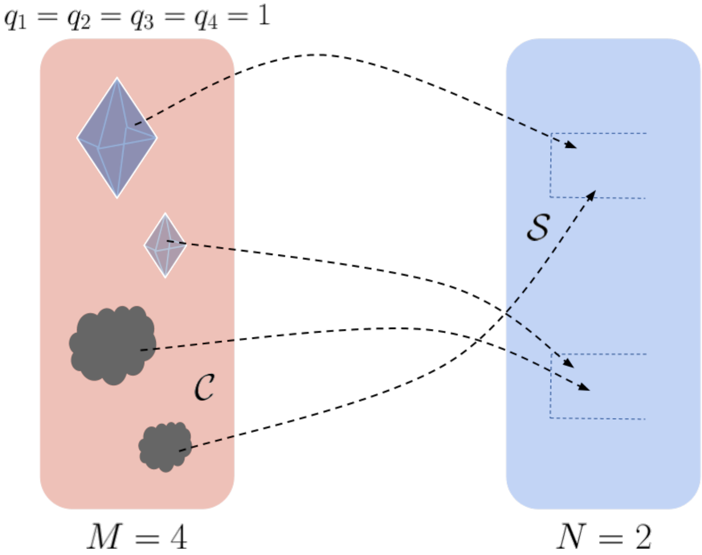


$M = 4$ $N = 2$

Example (Dictatorship)



Example (Approximate-CEEI)



Definition (Equilibrium)

Take an economy $\mathcal{E}$. For this economy, the allocation $\mathbf{x}$, budgets $\mathbf{b}$ and prices $\mathbf{p}$ constitute a (α, β)-Approximate CEEI if the following 3 conditions hold:

$$x_i = \arg\max_{x \in 2^{\mathcal{C}}} \{u_i(x) \mid \mathbf{p}\ x \leq b_i\} \quad \forall i \in \mathcal{S}$$

$$\alpha \geq \|z_1(\mathbf{p}), z_2(\mathbf{p}), \ldots, z_M(\mathbf{p})\|_2$$

$$1 + \beta \geq \max_{i \in \mathcal{S}} b_i \geq 1 = \min_{i' \in \mathcal{S}} b_{i'}$$

where $z_j(\mathbf{p}) = \sum_{i \in \mathcal{S}} x_{i \to j} - q_j$.

Theorem (Existence)

$$k = \max_{i \in \mathcal{S}} \max_{x \in \Psi} |x|$$

$$\sigma = \min \{2 \cdot k, M\}$$

For any $\beta > 0$, there exists a $(\sqrt{\sigma \cdot M}/2, \beta)$-Approx CEEI.

For any $\beta > 0$, any budget vector $\mathbf{b}$ and any $\varepsilon > 0$, there exists a $(\sqrt{\sigma \cdot M}/2, \beta)$-Approx CEEI with budgets $\mathbf{b}^*$ that satisfy the condition:

$$\varepsilon > |b_i^* - b_i| \quad \forall i \in \mathcal{S}$$

Proof (Existence)

Step 1: Understand $\sqrt{\sigma \cdot M}/2$ term.

Step 2: Convexify.

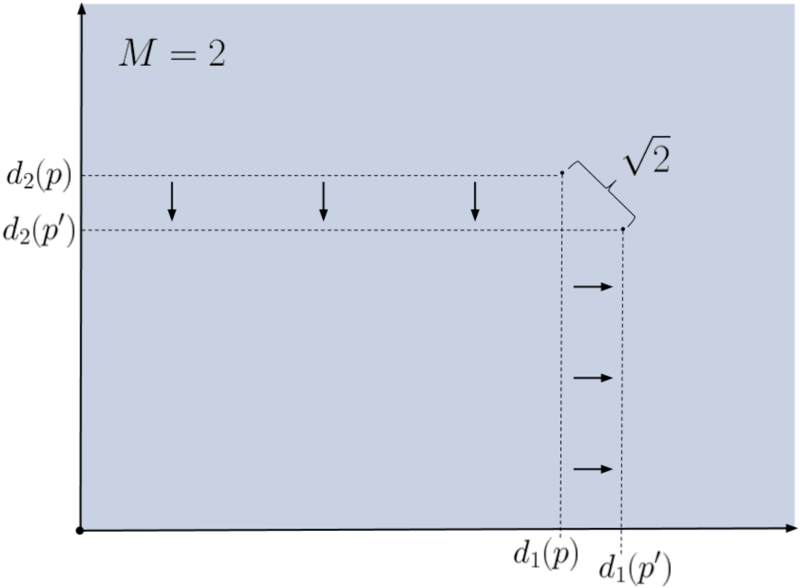
Step 3: Use fixed point theorem.

Proof (Existence, Step 1)

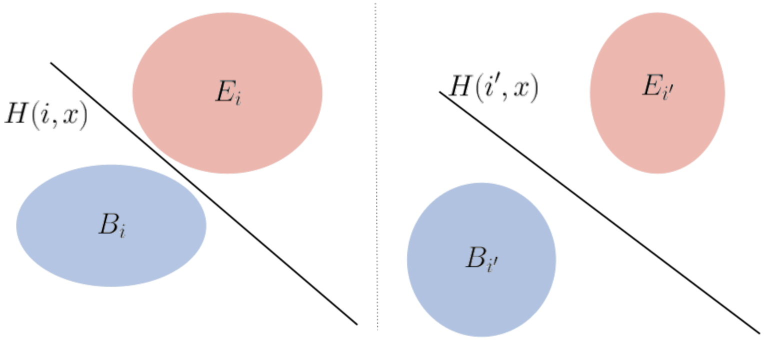
Bound demand discontuity (finiteness): Change in prices makes agent change entire $x_i \Rightarrow \sqrt{\sigma}$ change in direction of agg. demand.

Bound affected agents (inequality): $\leq M$ agents affected by change in prices.

Proof (Existence, Step 1)



Proof (Existence, Step 1)



Small change in prices $\Rightarrow M \cdot \sqrt{\sigma}$ change in demand.

Proof (Existence, Step 2)

Consider tâtonnement process:

$$f(\mathbf{p}) = \mathbf{p} + \mathbf{z}(\mathbf{p})$$

Fixed point of $f(\cdot)$ would be CE price vector.

Consider:

$$F(\mathbf{p}) = \text{co}\,\{\mathbf{y} \mid \exists \mathbf{p}_n \to \mathbf{p} \text{ s.t. } f(\mathbf{p}_n) \to \mathbf{y}\}$$

Proof (Existence, Step 3)

Cromme and Diener (1991): $F(\cdot)$ is upper-hemicontinuous.

Apply Kakutani's fixed point theorem: There exists a $\mathbf{p}$ such that $\mathbf{p} \in F(\mathbf{p})$.

i.e., $\mathbf{p} \in F(\mathbf{p})$ means that there exist a price vector arbitrarily close to $\mathbf{p}$ such that the convex combination of demand vectors exactly clears the market.

Proof (Existence, Step 3)

Cromme and Diener (1991): Let $P \subset \mathbb{R}^n$ be compact and convex and $f : P \mapsto P$ be any mapping, then we have that:

$$\alpha \geq \|f(\mathbf{p}) - \mathbf{p}\|$$

From intuition above, we know that $\sqrt{\sigma} \cdot M > \alpha$. Rest of proof moves from $\sqrt{\sigma} \cdot M$ to $\sqrt{\sigma \cdot M}/2$ bound.

Wish List

Equilibrium Concept:

[X] Existence

[ ] Efficiency

[ ] Fairness

Mechanism:

[ ] Strategy Proof-ness

Theorem (Efficiency)

Suppose $(\mathbf{x}, \mathbf{b}, \mathbf{p})$ is an (α, β)-Approx CEEI of the economy $\mathcal{E}$. Then, the allocation $\mathbf{x}$ is a Pareto efficient allocation in $\mathcal{E}$.

Proof (Efficiency)

Suppose $\mathbf{x}'$ Pareto improves $\mathbf{x}$.

By definition with strict preferences, if $x_i' \neq x_i$ then $\mathbf{p}\ x_i' > \mathbf{p}\ x_i$.

Thus, $\sum_{i=1}^{N} \mathbf{p}\ x_i' > \sum_{i=1}^{N} \mathbf{p}\ x_i$ which is a contradiction.

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Definition (Envy Bounded by a Single Good)

An allocation $\mathbf{x}$ satisfie envy bounded by a single good if, for any $i, i' \in \mathcal{S}$, either:

$$u_i(x_i) \geq u_i(x_{i'})$$

...or, there exists some good $j \in x_{i'}$ such that

$$u_i(x_i) \geq u_i(x_{i'} \setminus \{j\})$$

Theorem (Fairness)

For any economy $\mathcal{E}$, if $(\mathbf{x}, \mathbf{b}, \mathbf{p})$ is an (α, β)-Approx CEEI with

$$\beta < \frac{1}{k-1}$$

...then $\mathbf{x}$ satisfies the condition of envy bounded by a single good.

Proof (Fairness)

Suppose contradiction. Let $k' \leq k$ be number of objects envied in bundle x_i:

$$u_i(x_{i'} \setminus \{j_1\}) > u_i(x_i)$$
$$\vdots$$
$$u_i(x_{i'} \setminus \{j_{k'}\}) > u_i(x_i)$$

Agent i cannot afford any of these bundles:

$$b_{i'} \geq \mathbf{p}\,(x_{i'} \setminus \{j_1\}) > b_i$$
$$\vdots$$
$$b_{i'} \geq \mathbf{p}\,(x_{i'} \setminus \{j_{k'}\}) > b_i$$

Proof (Fairness)

By definition:

$$\begin{aligned} b_{i'} &\geq \mathbf{p}\ x_{i'} \\ &\geq p_1 + p_2 + \cdots + p_{k'} \end{aligned}$$

By adding in envy definition:

$$\begin{aligned} (k' - 1) \cdot b_{i'} \geq (k' - 1) \cdot (&\mathbf{p}\ x_{i'}) > k' \cdot b_i \\ \frac{b_{i'}}{b_i} &\geq \frac{k'}{k' - 1} \\ &\geq \frac{k}{k - 1} \end{aligned}$$

$(k - 1)^{-1} > \beta$ gives contradiction!

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Definition (Approx. CEEI Mechanism)

1. Each agent i reports her utility function $\hat{u}_i$.
2. Check for $(0, 0)$-Approx. CEEI's.
3. If non-empty:
   - Choose random $(\mathbf{x}, \mathbf{b}, \mathbf{p})$.
4. If empty:
   - Choose target budget $\mathbf{b}'$ uniformly from $[1, 1 + \beta]$ with $\beta < \min\left\{N^{-1}, (k-1)^{-1}\right\}$.
   - Set $\varepsilon \approx 0$, $\delta < 1 - N \cdot \beta$ and $\alpha \leq \sqrt{\sigma \cdot M}/2$.
   - Compute set of feasible (α, β)-Approx. CEEI's.
   - Choose random $(\mathbf{x}, \mathbf{b}, \mathbf{p})$ from set with minimum α and $\|\mathbf{b} - \mathbf{b}'\|$ small.

Definition (Continuum Replication)

The continuum replication of an economy $\mathcal{E}$ written as:

$$\mathcal{E}^{\infty} = \left(\mathcal{S}^{\infty}, \mathcal{C}, (q_j)_{j=1}^{M}, (\Psi_i)_{i \in \mathcal{S}^{\infty}}, (u_i)_{i \in \mathcal{S}^{\infty}} \right) \quad (1)$$

can be constructed as by replacing each agent in the original economy with a unit mass of identical agents.

$\mathcal{S}^{\infty} = (0, N]$ so that agent 1 is replaced with the mass $(0, 1]$, agent 2 with the mass $(1, 2]$, agent 3 with the mass $(2, 3]$ and so on...

Theorem (Strategy Proof-ness)

A mechanism is strategy proof in the large if it is exactly strategy proof in the continuum replication of any finite economy.

The approximate CEEI mechanism is strategy proof in the large.

Proof (Strategy Proof-ness)

Pick an economy $\mathcal{E}$ and consider its continuum replication $\mathcal{E}^\infty$.

Consider agent $i \in \mathcal{S}^\infty$ and fix all other agent's reports.

Agent i has measure 0 so cannot affect prices.

By definition of approximate-CEEI, agent i does best by truth telling given budget b_i.

End