Credible Monetary Policy in an Infinite Horizon Model: Recursive Approaches

by Roberto Chang, JET 1998
presented by Taisuke Nakata

October 20th, 2010

Introduction

This paper develops a recursive method to study optimal policy in a dynamic model.

- Ramsey policy
- Sustainable policy

Two key ingredients:

- Finding appropriate state variables.
- Defining an appropriate operator.

Environment

- A representative household:
  - $\sum_{t=0}^{\infty} \beta^t \left[u(c_t) + v(m_t^H) \right]$, $m_t^H \equiv q_t M_t^H$.
  - HBC: $c_t + q_t M_t^H \leq y_t - x_t + q_t M_{t-1}^H$, $m_t^H \in [0, \bar{m}] \equiv M$.
  - $y_t = f(x_t)$
  - $M_{-1}^H = M_{-1}$ given.
- Government:
  - GBC: $-x_t = q_t(M_t^G - M_{t-1}^G)$
  - $M_{-1}^G = M_{-1}$ given.
  - $h_t \equiv M_{t-1}^G / M_t^G$
  - $h_t \in [\underline{h}, \bar{h}] \equiv H$ ($\Rightarrow x_t \in X$)
- Notation: $\mathbf{z} \equiv \{z_t\}_{t=0}^{\infty}$, $\mathbf{z}^t \equiv \{z_s\}_{s=0}^{t}$, and $\mathbf{z}_t \equiv \{z_s\}_{s=t}^{\infty}$ for any z.

A Competitive Equilibrium

Proposition: A competitive equilibrium is characterized by a sequence $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ such that, for all t, $m_t \in M$, $x_t \in X$, $h_t \in H$, and

$$\begin{aligned} x_t &= m_t(h_t - 1) \\ m_t \left[u'[f(x_t)] - v'(m_t) \right] &\leq \beta u'[f(x_{t+1})](m_{t+1} + x_{t+1}), \; (= \text{if } m_t < \bar{m}) \\ &\equiv \beta \theta_{t+1} \end{aligned}$$

θ_t: "promised marginal utility"

Notation: $CE \equiv \{(\mathbf{m}, \mathbf{x}, \mathbf{h}) | \text{the conditions above hold.}\}$

Ramsey Plan: A Standard Formulation

$$V^R = \max_{(\mathbf{m},\mathbf{x},\mathbf{h})\in CE} \sum_{t=0}^{\infty} \beta^t \left[u[f(x_t)] + v(m_t) \right]$$

Ramsey Plan: A Recursive Formulation

Let

$$\Omega \equiv \{\theta \in \mathbb{R} : \theta = u'[f(x_0)](m_0 + x_0) \text{ for some } (\mathbf{m}, \mathbf{x}, \mathbf{h}) \in CE\}$$

$$\Gamma(\theta) \equiv \{(\mathbf{m}, \mathbf{x}, \mathbf{h}) \in CE : \theta = u'[f(x_0)](m_0 + x_0)\}$$

We can write the Ramsey problem in two steps:

$$w^*(\theta) = \max_{(\mathbf{m},\mathbf{x},\mathbf{h})\in\Gamma(\theta)} \sum_{t=0}^{\infty} \beta^t \left[u[f(x_t)] + v(m_t)\right]$$

$$V^R = \max_{\theta\in\Omega} w^*(\theta)$$

Ramsey Plan: A Recursive Formulation

Proposition: $w^*(\theta)$ satisfies the functional equation:

$$
\begin{aligned}
w(\theta) &= \max_{(m,x,h,\theta')} u[f(x)] + v(m) + \beta w(\theta') \\
\text{s.t.} & \\
& (m,x,h,\theta') \in M \times X \times H \times \Omega \\
& \theta = u'[f(x)](m+x) \\
& x = m(h-1) \\
& m\big[u'[f(x)] - v'(m)\big] \leq \beta\theta', = \text{if } m < \bar{m}
\end{aligned}
$$

Conversely, if a bounded function $w : \Omega \to \mathbb{R}$ satisfies the above functional equation, then $w = w^*$

Ramsey Plan: Finding Ω

An operator on $\mathbb{B}$: For any $Q \subseteq \mathbb{R}$, define a new set $\mathbb{B}(Q)$ as follows.

$$
\begin{aligned}
\mathbb{B}(Q) \quad = \quad & \{\theta \in \mathbb{R} | \\
& \exists (m, x, h, \theta') \in M \times X \times H \times Q \text{ such that} \\
& \theta = u'[f(x)](m + x) \\
& x = m(h - 1) \\
& m\big[u'[f(x)] - v'(m)\big] \leq \beta\theta', = \text{if } m < \bar{m}\}
\end{aligned}
$$

Prop. 1: If $Q \subseteq \mathbb{B}(Q)$, then $\mathbb{B}(Q) \subseteq \Omega$

Prop. 2: $\Omega = \mathbb{B}(\Omega)$

Prop. 3: Let $Q_0 = [0, \bar{\theta}]$. Let $Q_n = \mathbb{B}(Q_{n-1})$. Then, $Q_n \supseteq Q_{n+1}$ and $\Omega = \cap_{n=0}^{\infty} Q_n$.

Sustainable Plan: Set-Up (1)

A government strategy, $\sigma \equiv \{\sigma_t\}_{t=0}^{\infty}$

- $\sigma_0 \in H$
- $\sigma_t : H^{t-1} \to H$ (i.e., $h_t = \sigma_t(\mathbf{h}^{t-1})$)

An allocation strategy, $\alpha \equiv \{\alpha_t\}_{t=0}^{\infty}$

- $\alpha_t : H^t \to M \times X$ (i.e., $(m_t, x_t) = \alpha_t(\mathbf{h}^t)$)

Remarks:

- (σ,α) induces an outcome $(\mathbf{m}, \mathbf{x}, \mathbf{h})$.
- (σ,α) implies a value, $w = \sum_{t=0}^{\infty} \beta^t \left[u[f(x_t)] + v(m_t)\right]$, and an initial promised MU, $\theta = u'[f(x_0)](m_0 + x_0)$.
- After any history $\mathbf{h}^{t-1}$, the continuation of a strategy is also a strategy.

Sustainable Plan: Set-Up (2)

Let $CE_H \equiv \{\mathbf{h} \in H^\infty | \exists(\mathbf{m}, \mathbf{x}) \text{ such that } (\mathbf{m}, \mathbf{x}, \mathbf{h}) \in CE\}$

Definition: σ is admissible if, after any history $\mathbf{h}^{t-1}$, $\mathbf{h}_t$ induced by the continuation of σ belongs to CE_H

Definition: Given σ, α is competitive if, after any history $\mathbf{h}^t$, the continuation of α and σ induce a $(\mathbf{m}, \mathbf{x}, \mathbf{h}) \in CE$.

Sustainable Plan: Definition

Definition: (σ,α) is a sustainable plan if

- σ is admissible.
- α is competitive given σ.
- After any history $\mathbf{h}^{t-1}$, the sequence $\mathbf{h}_t$ induced by σ maximizes the household's utility over CE_H given α.

Definition: $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ is a sustainable outcome if it is induced by a sustainable plan.

Remark: After any history $\mathbf{h}^{t-1}$, the continuation of a sustainable plan is itself a sustainable plan.

Sustainable Plan

$$
\begin{aligned}
S \quad \equiv \quad & \{(w,\theta) | \exists \text{ a sustainable outcome } (\mathbf{m},\mathbf{x},\mathbf{h}) \\
& \text{with value } w \text{ and } u'[f(x_0)](m_0 + x_0) = \theta\}
\end{aligned}
$$

Sustainable Plan: Finding S (1)

Take a sustainable plan with a particular value of w and θ.

$$
\begin{aligned}
(1) \qquad & w = u[f(x(h^*))] + v[m(h^*)] + \beta w'(h^*) \\
& \geq u[f(x(h))] + v[m(h)] + \beta w'(h) \ \forall h \\
(2) \qquad & \theta = u'[f(x(h^*))](m(h^*) + x(h^*)) \\
(3) \qquad & x(h) = m(h)(h-1) \ \forall h \\
(4) \qquad & m(h)\left[u'[f(x(h))] - v'(m(h))\right] \leq \beta\theta'(h) \ (,= \text{ if } m(h) < \bar{m}) \ \forall h
\end{aligned}
$$

$(w'(h), \theta'(h))$ must belong to S for all h.

Sustainable Plan: Finding S (2)

An operator $\mathbb{D}$: For any set $Z \subseteq \mathcal{W} \times \Omega$, define a new set $\mathbb{D}(Z)$ as follows.

$$\begin{aligned} \mathbb{D}(Z) \quad = \quad & \{(w, \theta) | \\ & \exists\ h^* \text{ and} \\ & (m(h), x(h), w'(h), \theta'(h)) \in M \times X \times Z \text{ for all h} \\ & \text{such that (1), (2), (3), and (4) hold.}\} \end{aligned}$$

Prop. 1: If $Z \subseteq \mathbb{D}(Z)$, then $\mathbb{D}(Z) \subseteq S$

Prop. 2: $S = \mathbb{D}(S)$

Prop. 3: Let $Z_0 = \mathcal{W} \times \Omega$, and $Z_n = \mathbb{D}(Z_{n-1})$. Then, $S = \cap_{n=0}^{\infty} Z_n$.