Equilibrium Selection in global games with strategic complementarities
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Discussion by Michal Szkup

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November 2010
Global Games

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- Let $\Lambda = \{1, .., l\}$ be the set of players;

In the incomplete information version of the above game, called $G(v)$:

- $\theta$ is drawn from a continuous distribution $\Phi$;
- Each player observes $x_i = \theta + v \eta_i$, a noisy signal of $\theta$;
- $\eta_i$ is distributed according to atomless distribution $f_i$ with bounded support;

We refer to the above game $G(v)$ as a global game.
Consider the following complete information game $\Gamma(\theta)$

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  - the action that survives is independent of the distribution of the noise
Frankel Morris and Pauzner (2003)

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Focus on the games with strategic complementarities;
Assumptions

(A1) Strategic Complementarities:

If \( a_i \geq a_i' \) and \( a_{-i} \geq a_{-i}' \) then \( \forall \theta, \Delta u_i (a_i, a_i', a_{-i}, \theta) \geq \Delta u_i (a_i, a_i', a_{-i}', \theta) \)
Assumptions

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   If \( a_i \geq a'_i \) and \( a_{-i} \geq a'_{-i} \) then \( \forall \theta, \Delta u_i (a_i, a'_i, a_{-i}, \theta) \geq \Delta u_i (a_i, a'_i, a'_{-i}, \theta) \)

2. **(A2) Dominance regions**

   \( \exists \theta \) such that \( \forall \theta < \bar{\theta} \) and \( \forall a'_i \neq 0, \Delta u (0, a'_i, a_{-i}, \theta) > 0 \)

   \( \exists \bar{\theta} \) such that \( \forall \theta > \bar{\theta} \) and \( \forall a'_i \neq 1, \Delta u (1, a'_i, a_{-i}, \theta) > 0, 0 \bar{\theta} > \theta \)
Assumptions

1. (A1) Strategic Complementarities:

If $a_i \geq a'_i$ and $a_{-i} \geq a'_{-i}$ then $\forall \theta, \Delta u_i (a_i, a'_i, a_{-i}, \theta) \geq \Delta u_i (a_i, a'_i, a'_{-i}, \theta)$

2. (A2) Dominance regions

$\exists \theta$ such that $\forall \theta < \theta$ and $\forall a'_i \neq 0$, $\Delta u (0, a'_i, a_{-i}, \theta) > 0$

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3. (A3) State monotonicity:

$\exists K_0$ such that $\forall a_i \geq a'_i$ and $\theta, \theta' \in [\theta, \bar{\theta}]$ , $\theta \geq \theta'$ we have

$\Delta u (a_i, a'_i, a_{-i}, \theta) - \Delta u (a_i, a'_i, a_{-i}, \theta') \geq K_0 (a_i - a'_i) (\theta - \theta')$
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4. (A4) Payoff continuity

   $u_i (a_i, a_{-i}, \theta)$ is continuous in all arguments
Strategies

- A pure strategy of a player $i$ is a function $s_i : \mathbb{R} \rightarrow A_i$.
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- A profile $s'$ is **higher** than $s$ ($s' \succeq s$) if $s'_i(x_i) \geq s_i(x_i)$ for all $x_i$
- A mixed strategy is a probability distribution over pure strategies
Theorem (1)

\[ \exists \text{ an increasing strategy profile } s^* \text{ such that if, for each } v > 0, s^v \text{ is a strategy profile that survives iterative deletion of strategies in } G(v), \text{ then } \]

\[ \lim_{v \to 0} s^v_i (x_i) = s^*_i (x_i) \]

for almost all \( x_i \).

This theorem states that as noise vanishes the iterative deletion of dominated strategies selects a unique Bayesian Nash Equilibrium of the game.
Intuition for Theorem 1 (Simple Case)

- 2 players with $A_i = [0, 1]$ and uniform prior
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- For any $v > 0$ use iterative deletion of dominated strategies to find bounds on the set of rationalizable strategies, $\underline{S}(v)$ and $\overline{S}(v)$. 

Assumptions $(A_1)$ and $(A_3)$ imply that $\underline{S}(v) = \overline{S}(v)$ and if $\delta > 0$ then $\underline{S}(x + \delta) > a$.
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$\begin{aligned}
 a^* &\geq \overline{S}(x^*) = \underline{S}(x^* + \delta) & \text{and if } \delta > 0 \underline{S}(x^* + \delta) > a^*
\end{aligned}$
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\[ a^* \geq \bar{S}(x^*) = \underline{S}(x^* + \delta) \] and if $\delta > 0$ $\underline{S}(x^* + \delta) > a^*$

$\implies \delta = 0$ and so $\bar{S} = \underline{S} = \tilde{S}$
Intuition for Theorem 1 (General Case)

Solve first a game in which agents have a uniform prior and their utility is given by \( u_i(a_i, a_{-i}, x_i) \).
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- Then show that as noise vanishes, the simplified game "converges" to the original game.
- More precisely, the set of rationalizable strategies of the simplified game and original game converges.
- This result holds for any prior and any finite number of players or continuum of players.
A partial characterization of equilibrium

We can characterize the surviving equilibria of the game when noise is small.

**Definition**

\( Q(\varepsilon, \nu) \) is the set of parameters \( \theta \) for which no Nash Equilibrium action profile \( a \) of the complete information game with payoffs \( (u_i(\cdot, \theta'))_{i=1}^l \) for some \( \theta' \in [\theta - \varepsilon, \theta + \varepsilon] \), such that for every strategy \( s^\nu \) surviving iterative deletion of dominated strategies in \( G(\nu) \), \( \forall i \quad |s^\nu_i(\theta) - a_i| < \varepsilon \)
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- Define \( Q(\varepsilon, \nu) \) be a set of \( \theta \) for which surviving strategy profiles in \( G(\nu) \) do not prescribe all players to play close to some pure strategy Nash equilibrium of a complete information game.
Theorem (2)

In \( G(v) \) in the limit as \( v \to 0 \) for almost all payoff parameter \( \theta \), players play arbitrary close to some pure strategy Nash Equilibrium of the complete information game with payoffs \( u(\cdot, \theta') \) that is arbitrary close to \( \theta \), i.e. \( \forall \epsilon > 0 \) there is \( \bar{v} \) s.t. \( \forall v < \bar{v}, |\theta' - \theta| < \epsilon \).
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Further results

Theorem (3)

There exists a two-person, four-action game satisfying (A1) – (A5) in which for different noise structure different equilibria are selected in the limit as the signal errors vanish.
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Theorem (4)

If the complete information game at some payoff parameter \( \theta \) is quasiconcave in \( a_i \) and has local potential maximizer (LP-maximizer) \( a^* \) then \( s^*(\theta) = a^* \) regardless of the noise structure.
Conclusions

- Limit Uniqueness holds for a general class of global games with strategic complementarities

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Appendix: Intuition for Theorem 1

Consider the following game:

- 2 players with symmetric actions
- $A_i = [0, 1]$ for $i = 1, 2$
- $\theta$ is distributed uniformly
- Rest of the assumptions hold
Appendix: Intuition for Theorem 1

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- rest is unchanged
- all the assumptions hold
Appendix: Step 1

- Assume that player $j$ plays according to a strategy $s^0(x_i) = 0$. 

Consider now player $i$. By assumption (A2), $\exists x_1$ such that if $x_i > x_1$ then $a_i(x_i) = 1$. Hence no player will ever choose a pure strategy that lies below $s^1(x_i) = 1$ if $x_i > x_1$ and $s^1(x_i) = 0$ otherwise.
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- Hence no player will ever choose a pure strategy that lies below $s^1$ where $s^1(x_i) = 1$ if $x_i \geq x^1$ and $s^1(x_i) = 0$ otherwise.
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![Diagram showing strategy choice for player i with \( a_i = 0 \) and \( a_i = 1 \) based on \( x_i \).]
Appendix: Step 2

- Assume that player $j$ plays according to a strategy $s_1^1$ as defined above.
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- Assume that player $j$ plays according to a strategy $s^1$ as defined above.
- By (A1) player $i$’s best response to $s^1$, call it $s^2$ is weakly above $s^1$. 

Denote the limit of this process as $S$. Similarly, denote by $S$ a strategy that survives in the limit iterated deletion of dominated strategies starting with $s^0(x_i) = 1$. 

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Equilibrium Selection in global games
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![Diagram](attachment:image.png)
Appendix: Final Step (1)

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- To do so define $\tilde{S}(x) = S(x + \delta)$ such that $\tilde{S} \geq \bar{S}$ and $\exists x$ s.t. $\tilde{S}(x) = \bar{S}(x)$
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- Call this signal $x^*$ and define $a^*$ as a best response of player $i$ who observed $x^*$ and believes $s_j = \tilde{S}$
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Call this signal $x^*$ and define $a^*$ as a best response of player $i$ who observed $x^*$ and believes $s_j = \tilde{S}$
- Since $\tilde{S} \geq \bar{S}$ it follows by (A1) that $a^* \geq \bar{S}(x^*) = S(x^* + \delta)$
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However, his estimate of $\theta$ is strictly higher at $x + \delta$ by (A3), it follows that $\delta > 0$.
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  - player \( i \) observes \( x^* \) and expects his opponent to play \( \tilde{S} \)
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- In both cases he faces the same distribution of action of his opponent.
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- In both cases he faces the same distribution of action of his opponent.
- However, his estimate of $\theta$ is strictly higher at $x^* + \delta$
- by (A3) it follows that $\forall \delta > 0 \ S(x^* + \delta) > a^* \implies \delta = 0$
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  - player $i$ observes $x^*$ and expects his opponent to play $\tilde{S}$
  - player $i$ observes $x^* + \delta$ and expects his opponent to play $S$

- In both cases he faces the same distribution of action of his opponent.

- However, his estimate of $\theta$ is strictly higher at $x^* + \delta$

- by (A3) it follows that $\forall \delta > 0 \ S (x^* + \delta) > a^* \implies \delta = 0$

- This last implication follows from the fact that we also have $a^* \geq S (x^* + \delta)$
Consider two cases:

- player \( i \) observes \( x^* \) and expects his opponent to play \( \tilde{S} \)
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That would lead to contradiction unless \( \tilde{S}(x) = S(x) \)
Definition (Potential Function)

Let $N$ be a finite set of players, $Y_i$ be the set of $i$'s strategies and $u_i : Y \rightarrow R$ be $i$'s payoff function. Then a function $P : Y \rightarrow R$ is called a potential function if for every $i$ and every $Y_{-i}$ we have

$$u(x, y_{-i}) - u(x', y_{-i}) = P(x, y_{-i}) - P(x', y_{-i})$$
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\[
    u(x, y_{-i}) - u(x', y_{-i}) = P(x, y_{-i}) - P(x', y_{-i})
\]

- A potential function is a common payoff function such that a change is player \( i \)'s payoff from switching action to another is always the same as the change in the potential function.
Appendix (Intuition for Theorem 4)

For simplicity assume:

- a game is symmetric and the set of action is finite
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- Near discontinuity points there still might be miscoordination → potential maximizing strategy is not played there.

Consider now iterative best response procedure:
- Iterative best responding increases the potential function away from discontinuities. Best response must coincide with $x^*_I$.
- This iterative procedure converges to equilibrium $x^*$ but $x^*$ is payoff maximizing profile, so as noise vanishes the best responses must be except for small neighborhood of discontinuity points.
- Hence as noise vanishes, players must play potential maximizing action.
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- A potential function is maximized away from discontinuities for small enough noise.
- Near discontinuity points there still might be miscoordination $\Rightarrow$ potential maximizing strategy is may not be played there.
- Consider now iterative best response procedure:

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Appendix (Intuition for Theorem 4)

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- A potential function is maximized away from discontinuities for small enough noise.
- Near discontinuity points, there might still be miscoordination, so the potential maximizing strategy is not played there.
- Consider the iterative best response procedure:
  - Best responding increases the potential function $\nu$.
  - Away from discontinuities, best response coincides with $a^*_x$. 
A potential function is maximized away from discontinuities for small enough noise.

Near discontinuity points there still might be miscoordination \( \implies \) potential maximizing strategy is may not be played there.

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Appendix (Intuition for Theorem 4)

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- Near discontinuity points there still might be miscoordination, meaning a potential maximizing strategy is may not be played there.
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