Risk sensitive allocations with multiple goods: existence, survivorship, and the curse of the linear approximation

Ric Colacito and Max Croce (2010, wp)

Håkon Tretvoll

Reading Group Presentation

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Risk-sharing results

- Time-separable preferences + single good
  $\Rightarrow$ allocations a time-invariant function of aggregate endowment
- Heterogeneous risk-sensitive preferences + single good
  $\Rightarrow$ wealth distribution evolves over time (Anderson, 2005)
- **This paper**: risk-sensitive preferences + two goods and home-bias
  $\Rightarrow$ interesting dynamics with equal risk-sensitivity + survival
- Also: first-order approximation does not capture dynamics of the economy (third-order does)

Setup

- Preferences: $\forall i \in \{1, 2\}$

$$U_i(c_i, q_i) = (1-\delta)\log c_i + \delta\theta \log \sum_{s'} \pi(s') \exp\left\{\frac{q_i(s')}{\theta}\right\}$$

- Consumption aggregators:

$$c_1 = (x_1)^{\alpha}(y_1)^{1-\alpha} \qquad c_2 = (x_1)^{1-\alpha}(y_1)^{\alpha}$$

where $\alpha > 1/2$ implies home-bias

- Endowments:
  - First-order Markov process
  - Finite set of values $\mathcal{N} = \{1, \ldots, n\}$

Solving the planner's problem

$$Q_p(s,\mu_1) = \max_{\{x_i,y_i,q_{i,s'}\}_{i\in\{1,2\},s'\in\mathcal{N}}} \sum_{i=1}^{2} \mu_i \left((1-\delta)\log c_i(s,\mu_1) + \delta\theta \log \sum_{s'} \pi(s') \exp\left\{ \frac{q_{i,s'}}{\theta} \right\} \right)$$

subject to

$$\mu_2 = 1 - \mu_1$$

$$0 \le x_1 \le X(s) \qquad 0 \le x_2 \le X(s) - x_1$$

$$0 \le y_1 \le Y(s) \qquad 0 \le y_2 \le Y(s) - y_1$$

$$c_i = (x_i)^{\nu_i} (y_i)^{1-\nu_i}, \qquad \forall i \in \{1,2\}$$

$$0 \le \min_{\mu_1(s')\in[0,1]} Q_p(s',\mu_1(s')) - \mu_1(s')q_{1,s'} - (1-\mu_1)q_{2,s'} \quad \forall s' \in \mathcal{N}$$

First-order conditions

- W.r.t. allocations:

$$\frac{\partial \log c_1}{\partial x_1} \cdot \phi = \frac{\partial \log c_2}{\partial x_2} \qquad \frac{\partial \log c_1}{\partial y_1} \cdot \phi = \frac{\partial \log c_2}{\partial y_2}$$

$$X = x_1 + x_2 \qquad Y = y_1 + y_2$$

- W.r.t. continuation utilities:

$$\phi' = \phi \cdot \mathcal{M}(s', \phi') \Rightarrow \ \phi = f_\phi(s', \phi)$$

where

$$\mathcal{M}(s', \phi') \equiv \frac{\exp\{U_1(s', \phi')/\theta\}}{\sum_{s'} \pi(s') \exp\{U_1(s', \phi')/\theta\}} \frac{\sum_{s'} \pi(s') \exp\{U_2(s', \phi')/\theta\}}{\exp\{U_2(s', \phi')/\theta\}}$$

- Note that $\phi = \mu_1/(1-\mu_1)$ and $U_i(s, \phi)$ is agent i's utility at the optimum

An example

- To illustrate results they provide plots from the solution of the following model:

Parameter	Value	State, s	$\pi(s)$	$X(s)$	$Y(s)$
δ	0.95	1	$1/4$	103	103
α	0.98	2	$1/4$	103	100
γ	25	3	$1/4$	100	103
θ	$\frac{1}{-24}$	4	$1/4$	100	100

Proving the main result

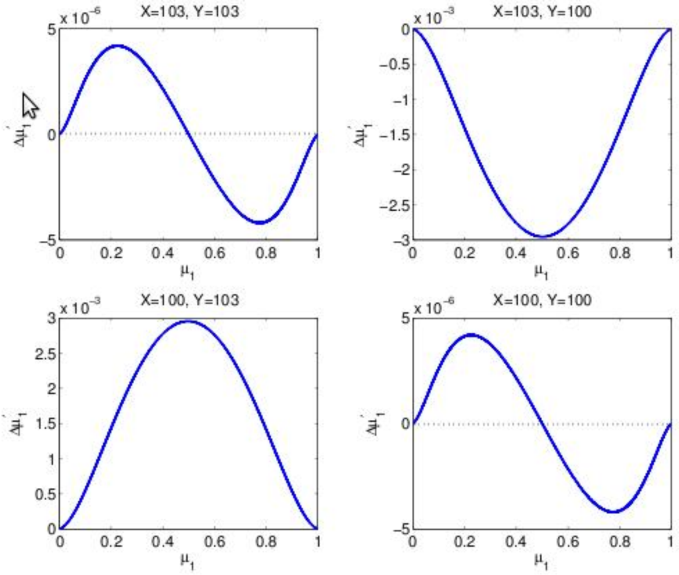
- **Lemma** 1:
  The ratio of Pareto weights can be decomposed as:

$$\mathbb{E}[\phi'|\phi] = \phi - \frac{\text{cov}\left(\exp\{U_2'/\theta\}, \phi' \,|\phi\right)}{\mathbb{E}\left(\exp\{U_2'/\theta\}|\phi\right)}$$

- **Lemma** 2:
  Let events $\{a, b\} \in \mathcal{N}$ be such that $X(a)/Y(a) > X(b)/Y(b)$. Then the ratio of Pareto weights is such that

$$\phi(a) < \phi(b)$$

  If $X(a)/Y(a) = X(b)/Y(b)$, then $\phi(a) = \phi(b)$.


X=103, Y=103
x 10^-6
Δμ1'
μ1
X=103, Y=100
x 10^-3
Δμ1'
μ1
X=100, Y=103
x 10^-3
Δμ1'
μ1
X=100, Y=100
x 10^-6
Δμ1'
μ1

Some definitions/assumptions

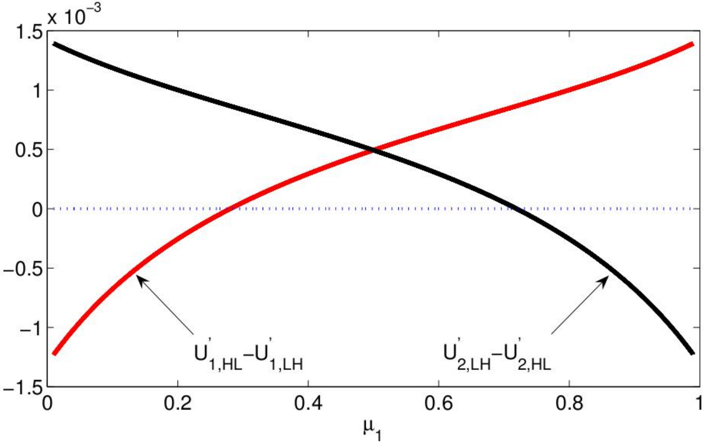
- **Balanced endowment space** (assn.):
  Let the support of the endowment of good X be given by the vector $Z = [z_1, z_2, \ldots, z_N]$. Let the support of the endowment of good Y be Z as well. The endowments of the two goods takes values in the finite set $\mathcal{N}$ given by all the possible pairwise permutations of Z. We refer to $\mathcal{N}$ as a balanced endowment space.

- **Symmetric States** (defn.):
  Let the states $s_i, s_{-i} \in \mathcal{N}$ be such that $s_i = \{X_i = X(i), Y_i = Y(i)\}$ and $s_{-i} = \{X_{-i} = Y(i), Y_{-i} = X(i)\}$. Then s_i and s_{-i} are symmetric states.

- **Symmetric Probabilities** (assn.):
  Let the states $s_i, s_{-i} \in \mathcal{N}$ be symmetric. Then $\pi(s_i) = \pi(s_{-i})$.

- **Lemma** 3:
For any two symmetric states s_i and s_{-i}, such that $X(s_i) > Y(s_i)$, there exists a finite $\tilde{\phi}_2^i > 1$ such that $U_2\left(s_i, f_\phi\left(s_i, \tilde{\phi}_2^i\right)\right) = U_2\left(s_{-i}, f_\phi\left(s_{-i}, \tilde{\phi}_2^i\right)\right)$, and $U_2\left(s_i, f_\phi\left(s_i, \phi_2^i\right)\right) > U_2\left(s_{-i}, f_\phi\left(s_{-i}, \phi_2^i\right)\right), \forall \phi_2^i > \tilde{\phi}_2^i$.

- **Corollary**:
Similar result for agent 1.


x 10^-3
1.5
1
0.5
0
−0.5
−1
−1.5
0
0.2
0.4
0.6
0.8
1
μ1
U'1,HL−U'1,LH
U'2,LH−U'2,HL

- **Covariance of symmetric states**:
Let $s_i, s_{-i} \in \mathcal{N}$ be symmetric states. The conditional covariance between two random variables h and g values on $\{s_i, s_{-i}\}$ is

$$\mathrm{cov}_{i,-i}[h, g|\phi] = \sum_{l=\{i,-i\}} p(s_l)h(s_l)g(s_l) - \left(\sum_{l=\{i,-i\}} p(s_l)h(s_l)\right)\left(\sum_{l=\{i,-i\}} p(s_l)g(s_l)\right)$$

where $p(s_l) \equiv \pi(s_l)/(\pi(s_i) + \pi(s_{-i}))$.

- **Lemma** 4:
Let $\overline{\mathrm{cov}}$ be the sum of the conditional covariances between $\exp\{U_2'/\theta\}$ and ϕ' across all symmetric states:

$$\overline{\mathrm{cov}} = \sum_i \mathrm{cov}_{i,-i}[\exp\{U_2'/\theta\}, \phi' \mid \phi]$$

If $\overline{\mathrm{cov}} < 0$, then $cov[\exp\{U_2'/\theta\}, \phi' \mid \phi] < 0$.

Ergodic distribution of Pareto weights

- **Proposition** 1:
  The stochastic process $\phi \in [0, \min_i\{\tilde{\phi}_2^i\}]$ is a submartingale.
- **Lemma** 5:
  A bounded submartingale cannot converge almost surely to its lower bound.
- Pareto weight on agent 1 does not converge to zero w.p. 1. Symmetric argument for agent 2.
- **Proposition** 2:
  The stochastic processes μ_1 and μ_2 cannot converge almost surely to either 0 or 1.

First-order approximation