Model Uncertainty, Limited Market Participation, and Asset Prices

Cao, Wang and Zhang

October 13, 2010
Motivation: Limited Stock Market Participation

- Mankiw and Zeldes (1991)
  1. Only 27.6% HHs hold stocks (PSID 1984)
  2. Only 47.7% HHs with liquid assets > $100,000

More recent data on participation puzzle

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  % HH holding asset, SCF 2007

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- Lot of structure: CARA and normality.

Main results

1. Uncertainty dispersion can cause non-participation
2. Equity premium can be decomposed into a risk premium and an uncertainty premium
3. Limited participation produces lower equity premium, by reducing uncertainty premium
4. Diversification discount under limited participation

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The economy

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Initial endowment: $x > 0$ shares
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Stock return $r \sim N(\mu, \sigma^2)$
$\sigma$ is known.
Investors do not have precise estimate for $E(r)$.
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3. Each agent will have a different set of priors \( \{\mu + v\} \).
Multi-priors expected utility and model uncertainty aversion

\[
\min_{Q \in \mathcal{P}} \{ E_Q[u(W)] \} \tag{1}
\]

where

\[
\mathcal{P} \equiv \{ Q / E_Q[\ln(dQ/dP)] < \eta \}
\]

P is a reference probability distribution and \( \eta \) is a measure of uncertainty.

Kogan and Wang (2002): normal distribution with common known variance, if \( E_Q(r) = \mu + v \), then

\[
P \equiv \{ N(\mu + v, \sigma^2) / v^2 \leq \phi^2_i \}
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Kogan and Wang (2002): normal distribution with common known variance,

if \(E_Q(r) = \mu + \nu\), then \(P \equiv \{ N(\mu + \nu, \sigma^2) / \nu^2 \leq \phi_i^2 \} \tag{2}\)

where \(\mu\) is common to every agent.
Homogeneous CARA utility over final wealth, $u(W)$

$$u(W) = -e^{-\gamma W}$$
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Heterogeneity in levels of uncertainty,

$$\phi_i \sim U(S), \quad S \equiv [\bar{\phi} - \delta, \bar{\phi} + \delta]$$

where $\delta \leq \bar{\phi}$ measures the dispersion of uncertainty among investors.
Agent $i$ problem,

$$\max_{D_i} \min_{Q \in \mathcal{P}} E_Q[u(W_{1i})]$$

s.t.

$$W_{1i} = xP + D_i(r - P)$$
Portfolio choice

- Agent i problem,

$$\max_{D_i} \min_{Q \in \mathcal{P}} \mathbb{E}_Q[u(W_{1i})]$$

s.t.

$$W_{1i} = xP + D_i(r - P)$$

- Under normality and CARA utility,

$$\max_{D_i} \left\{ D_i(\mu - \text{sgn}(D_i)\phi_i - P) - \frac{\gamma\sigma^2}{2}D_i^2 \right\}$$  \hspace{1cm} (4)
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Under normality and CARA utility,

\[
\max_{D_i} \left\{ D_i(\mu - \text{sgn}(D_i)\phi_i - P) - \frac{\gamma\sigma^2}{2}D_i^2 \right\}
\]

Solving for the optimal holding in stock,

\[
D_i = \begin{cases} 
\frac{1}{\gamma\sigma^2}(\mu - \phi - P) & \text{if } \mu - P > \phi_i \\
0 & \text{if } -\phi_i \leq \mu - P \leq \phi_i \\
\frac{1}{\gamma\sigma^2}(\mu + \phi - P) & \text{if } \mu - P < -\phi_i 
\end{cases}
\]
Equilibrium Market Participation

- In equilibrium $\mu - P > 0$
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Full participation equilibrium

$$x = \int_{\phi_i \in S} \frac{(\mu - \phi_i - P)}{\gamma \sigma^2} \frac{1}{2\delta} d\phi_i = \frac{(\mu - \phi - P)}{\gamma \sigma^2}$$

$$\Rightarrow \mu - P = \gamma \sigma^2 x + \phi$$

Full participation if $\gamma \sigma^2 x > \delta$
Equilibrium Market Participation

- In equilibrium $\mu - P > 0$
- Full participation equilibrium

$$
x = \int_{\phi_i \in S} \frac{(\mu - \phi_i - P)}{\gamma \sigma^2} \frac{1}{2\delta} d\phi_i = \frac{(\mu - \bar{\phi} - P)}{\gamma \sigma^2}
$$

$$
\Rightarrow \mu - P = \gamma \sigma^2 x + \bar{\phi}
$$

Full participation if $\gamma \sigma^2 x > \delta$

- Limited participation equilibrium

$$
\mu - \phi^* - P = 0
$$

$$
x = \int_{\phi_i < \phi^*, \phi_i \in S} \frac{(\mu - \phi_i - P)}{\gamma \sigma^2} \frac{1}{2\delta} d\phi_i
$$

$$
\Rightarrow \phi^* = \bar{\phi} - \delta + 2\sigma \sqrt{\gamma \delta x} = \mu - P
$$

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Rewrite equity premium under limited participation as,

\[ \mu - P = \frac{\gamma \sigma^2 x}{\alpha} + \phi_p \]  

(6)

where

\[ \alpha = \frac{1}{2\delta}(\phi^* - (\bar{\phi} - \delta)) = \sigma \sqrt{\frac{\gamma x}{\delta}} \]  

(7)

is the proportion of investors holding stock and,

\[ \phi_p = \frac{1}{2}(\phi^* + \bar{\phi} - \delta) = \bar{\phi} - \delta + \sigma \sqrt{\gamma \delta x} \]  

(8)

is average level of uncertainty among market participants.

Proposition

Suppose that \( \gamma \sigma^2 x < \delta \). Then, there is limited market participation in equilibrium and,

\[ \frac{\partial \alpha}{\partial \delta} < 0, \quad \frac{\partial \phi_p}{\partial \delta} < 0, \quad \frac{\partial (\mu - P)}{\partial \delta} < 0 \]  

(9)
Market Participation and Equity Premium

- Rewrite equity premium under limited participation as,

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(9)
Figure 1. $x = 0.3, \sigma = 0.25, \mu = 1.06, \gamma = 1, \bar{\phi} = 0.0375$
Extensions:

1. Two stocks
   - Two stocks, and two factor portfolios
   - \( r_M = r_1 + r_2 \)
   - Diversification discount: \( P_M < P_1 + P_2 \)
   - With limited participation, price is determined by investors with less uncertainty for each firm. The price is lower for the bundle, because being less uncertain about one firm does not necessarily imply less uncertainty about the other.
Other results from the paper

Extensions:

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2. Two periods
   - Introduce motive for hedging demand: non-tradable wealth correlated with asset return at \( t=1 \).
   - Introduce public signal to affect \( P_1 \)
   - Without public signal, hedging demand dominates aversion to uncertainty: full participation.
   - With public signal, limited participation equilibrium exists.
Appendix 0: Stockholding puzzle

Max \( s_1, s_2 \) \( U(c_0) + \beta E_0[U(c_1)] \)

s.t.

\[
\begin{align*}
  c_0 & = s_1 + s_2 - Y_0 \\
  c_1 & = s_1 R_1 + s_2 R_2 + Y_1 \\
\end{align*}
\]

\( \Rightarrow \) FOCs:

\[
\begin{align*}
  U'(c_0) & = \beta E_0 [R_1 U'(c_1)] \\
  U'(c_0) & = \beta E_0 [R_2 U'(c_1)] \\
\end{align*}
\]

\( \Rightarrow \)

\[
0 = E_0 [(R_1 - R_2) U'(c_1)]
\]
Applying Cov and BC, assuming $R_2$ risk free,

$$0 = (E_0[R_1] - R_2)E_0[U'(c_1)] + Cov_0[R_1U'(s_1R_1 + s_2R_2 + Y_1)]$$

if $s_1 = 0$ is solution,

$$0 = (E_0[R_1] - R_2)E_0[U'(s_2R_2 + Y_1)] + Cov_0[R_1U'(s_2R_2 + Y_1)]$$

assuming $Cov(Y_1, R_1) = 0$,

$$0 = (E_0[R_1] - R_2)E_0[U'(s_2R_2 + Y_1)]$$

With positive E.P. $(E_0[R_1] - R_2 > 0)$ and no bliss point $(E_0[U'(c_1)] > 0)$ this is a contradiction.
Appendix 1: proof Proposition 1

**Proposition**

Suppose that $\gamma \sigma^2 x < \delta$. Then, there is limited market participation in equilibrium and,

$$\frac{\partial \alpha}{\partial \delta} < 0, \quad \frac{\partial \phi_p}{\partial \delta} < 0, \quad \frac{\partial (\mu - P)}{\partial \delta} < 0$$  \hspace{1cm} (10)

$$\frac{\partial (\mu - P)}{\partial \delta} = \frac{\partial \gamma \sigma^2 x}{\partial \delta} + \frac{\partial \phi_p}{\partial \delta}$$

where

$$\frac{\partial \gamma \sigma^2 x}{\partial \delta} = -\delta \frac{\partial \alpha}{\partial \delta} = \delta^{-1/2} \sigma \sqrt{\gamma x^2}$$

$$\frac{\partial \phi_p}{\partial \delta} = \delta^{-1/2} \sigma \sqrt{\gamma x^2} - 1$$

$$\Rightarrow$$

$$\frac{\partial (\mu - P)}{\partial \delta} = \delta^{-1/2} \sigma \sqrt{\gamma x} - 1 < 0, \quad \text{if } \gamma \sigma^2 x < \delta$$
Three possible values: $w_1 = 0$, $w_2 = 10$, $w_3 = 20$

$\sigma^2 = 25$

Different probability distributions,

<table>
<thead>
<tr>
<th>$\pi_1$</th>
<th>$\pi_2$</th>
<th>$\pi_3$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1000</td>
<td>0.7472</td>
<td>0.1528</td>
<td>10.5279</td>
</tr>
<tr>
<td>0.2000</td>
<td>0.7325</td>
<td>0.0675</td>
<td>8.6754</td>
</tr>
<tr>
<td>0.3000</td>
<td>0.6746</td>
<td>0.0254</td>
<td>7.2540</td>
</tr>
<tr>
<td>0.4000</td>
<td>0.5944</td>
<td>0.0056</td>
<td>6.0557</td>
</tr>
<tr>
<td>0.5000</td>
<td>0.5000</td>
<td>0</td>
<td>5.0000</td>
</tr>
<tr>
<td>0.6000</td>
<td>0.3954</td>
<td>0.0046</td>
<td>4.0455</td>
</tr>
<tr>
<td>0.7000</td>
<td>0.2832</td>
<td>0.0168</td>
<td>3.1678</td>
</tr>
<tr>
<td>0.8000</td>
<td>0.1649</td>
<td>0.0351</td>
<td>2.3509</td>
</tr>
<tr>
<td>0.9000</td>
<td>0.0416</td>
<td>0.0584</td>
<td>1.5836</td>
</tr>
</tbody>
</table>
Appendix 3: Some evidence

Figure 2
Stock market participation and total annual stock returns for France, Germany, Italy, the Netherlands, Sweden, the United Kingdom, and the United States
Two Stocks

- Two stocks, and two factor portfolios

\[
\begin{pmatrix}
  r_1 \\
  r_2
\end{pmatrix} = \beta \begin{pmatrix}
  f_A \\
  f_B
\end{pmatrix} + \begin{pmatrix}
  \epsilon_1 \\
  \epsilon_2
\end{pmatrix}
\]

\[r_M = r_1 + r_2\]

- Diversification discount: \( P_M < P_1 + P_2 \)

- With limited participation, a firm’s pre-merger price is determined by investors with less uncertainty for that firm. After the merger investors have to buy both firms as a bundle, but being less uncertain about one firm does not necessarily imply less uncertainty about the other, so they offer a lower price for the combined firm.
Two periods

- Introduce motive for hedging demand: non-tradable wealth correlated with asset return at $t=1$.
- Introduce public signal to affect $P_1$
- Without public signal, hedging demand dominates aversion to uncertainty: full participation.
- With public signal, additional uncertainty about $P_1$. Under some parameters, limited participation.
Appendix 6: Static Ellsberg(1961)’s Experiment

- Challenge Savage(1954)’s expected utility with subjective probabilities framework.
- 30 balls in an urn, 10 are yellow, and 20 are either blue or green.
- Two choice problems:

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<th>Blue</th>
<th>Green</th>
</tr>
</thead>
<tbody>
<tr>
<td>Choice I</td>
<td>$f_1$</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$f_2$</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>Choice II</td>
<td>$f_3$</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$f_4$</td>
<td>0</td>
<td>4</td>
</tr>
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- Picking $f_1, f_4$ denotes ambiguity (or uncertainty) aversion, picking $f_2, f_3$ denotes love for ambiguity.
- Both picks violate Savage’s axioms and denote an attitude towards uncertainty.