Model Uncertainty, Limited Market Participation, and Asset Prices

Cao, Wang and Zhang

The Review of Financial Studies (2005)

October 13, 2010

Motivation: Limited Stock Market Participation

- Mankiw and Zeldes (1991)
  1. Only 27.6% HHs hold stocks (PSID 1984)
  2. Only 47.7% HHs with liquid assets $> \$100,000$

Motivation: Limited Stock Market Participation

- Mankiw and Zeldes (1991)
  1. Only 27.6% HHs hold stocks (PSID 1984)
  2. Only 47.7% HHs with liquid assets $> \$100,000$
- More recent data on *participation puzzle*

% **HH holding asset, SCF 2007**

Percentile of Income	Direct Stockholding	Direct and Indirect
All Families	17.9	51.1
Less than 20	5.5	13.6
40-60	14.0	49.5
90-100	47.5	91.0

Notes: Indirect Stockholdings are those in investment trusts, retirement accounts and other managed assets.

Motivation: Limited Stock Market Participation

- Mankiw and Zeldes (1991)
  1. Only 27.6% HHs hold stocks (PSID 1984)
  2. Only 47.7% HHs with liquid assets $> \$100,000$
- More recent data on *participation puzzle*

% **HH holding asset, SCF 2007**

Percentile of Income	Direct Stockholding	Direct and Indirect
All Families	17.9	51.1
Less than 20	5.5	13.6
40-60	14.0	49.5
90-100	47.5	91.0

Notes: Indirect Stockholdings are those in investment trusts, retirement accounts and other managed assets.

- The Problem: Limited Stock Market Participation difficult to reconcile with expected utility framework and positive equity premium.

Some related literature

- Vissing-Jorgensen (2002), and many others, explained limited participation by a fixed entry cost (i.e. cognitive, or transaction costs) and some heterogeneity.

Some related literature

- Vissing-Jorgensen (2002), and many others, explained limited participation by a fixed entry cost (i.e. cognitive, or transaction costs) and some heterogeneity.
- Haliassos and Bertaut (1995), using both calibration exercise and econometric analysis, showed that inertial factors and fixed costs, business cycle risks with short sale constraints, and departures from expected utility maximization are plausible explanations

Some related literature

- Vissing-Jorgensen (2002), and many others, explained limited participation by a fixed entry cost (i.e. cognitive, or transaction costs) and some heterogeneity.
- Haliassos and Bertaut (1995), using both calibration exercise and econometric analysis, showed that inertial factors and fixed costs, business cycle risks with short sale constraints, and departures from expected utility maximization are plausible explanations
- Ang, Bekaert and Liu (2002) studied disappointment aversion, and Dow and Costa Werlang (1992) focused on model uncertainty in partial equilibrium setup with representative agent.

This paper

- Analyze participation decision in general equilibrium, with model uncertainty and heterogeneous agents.

This paper

- Analyze participation decision in general equilibrium, with model uncertainty and heterogeneous agents.
- Combine multi-priors Knightian approach, and investors with heterogeneous levels of model uncertainty.

This paper

- Analyze participation decision in general equilibrium, with model uncertainty and heterogeneous agents.
- Combine multi-priors Knightian approach, and investors with heterogeneous levels of model uncertainty.
- Lot of structure: CARA and normality.

This paper

- Analyze participation decision in general equilibrium, with model uncertainty and heterogeneous agents.
- Combine multi-priors Knightian approach, and investors with heterogeneous levels of model uncertainty.
- Lot of structure: CARA and normality.
- Main results

This paper

- Analyze participation decision in general equilibrium, with model uncertainty and heterogeneous agents.
- Combine multi-priors Knightian approach, and investors with heterogeneous levels of model uncertainty.
- Lot of structure: CARA and normality.
- Main results
  1. Uncertainty dispersion can cause non-participation

This paper

- Analyze participation decision in general equilibrium, with model uncertainty and heterogeneous agents.
- Combine multi-priors Knightian approach, and investors with heterogeneous levels of model uncertainty.
- Lot of structure: CARA and normality.
- Main results
    1. Uncertainty dispersion can cause non-participation
    2. Equity premium can be decomposed into a risk premium and an uncertainty premium

This paper

- Analyze participation decision in general equilibrium, with model uncertainty and heterogeneous agents.
- Combine multi-priors Knightian approach, and investors with heterogeneous levels of model uncertainty.
- Lot of structure: CARA and normality.
- Main results
    1. Uncertainty dispersion can cause non-participation
    2. Equity premium can be decomposed into a risk premium and an uncertainty premium
    3. Limited participation produces lower equity premium, by reducing uncertainty premium

This paper

- Analyze participation decision in general equilibrium, with model uncertainty and heterogeneous agents.
- Combine multi-priors Knightian approach, and investors with heterogeneous levels of model uncertainty.
- Lot of structure: CARA and normality.
- Main results

  1. Uncertainty dispersion can cause non-participation
  2. Equity premium can be decomposed into a risk premium and an uncertainty premium
  3. Limited participation produces lower equity premium, by reducing uncertainty premium
  4. *Diversification discount* under limited participation

The economy

- Simplest case (1): 1-period static problem

The economy

- Simplest case (1): 1-period static problem
- Continuum of agents invest and consume

The economy

- Simplest case (1): 1-period static problem
- Continuum of agents invest and consume
- Timing: Portfolio choice, then consumption

The economy

- Simplest case (1): 1-period static problem
- Continuum of agents invest and consume
- Timing: Portfolio choice, then consumption
- Simplest case (2): one stock and one risk-free bond

The economy

- Simplest case (1): 1-period static problem
- Continuum of agents invest and consume
- Timing: Portfolio choice, then consumption
- Simplest case (2): one stock and one risk-free bond
- Initial endowment: $x > 0$ shares

The economy

- Simplest case (1): 1-period static problem
- Continuum of agents invest and consume
- Timing: Portfolio choice, then consumption
- Simplest case (2): one stock and one risk-free bond
- Initial endowment: $x > 0$ shares
- Risk-free rate 0

The economy

- Simplest case (1): 1-period static problem
- Continuum of agents invest and consume
- Timing: Portfolio choice, then consumption
- Simplest case (2): one stock and one risk-free bond
- Initial endowment: $x > 0$ shares
- Risk-free rate 0
- Stock return $r \sim N(\mu, \sigma^2)$
  1. σ is known.
  2. Investors do not have precise estimate for $E(r)$.
  3. Each agent will have a different set of priors $\{\mu + v\}$.

Preferences (1): multi-priors and uncertainty aversion

- Multi-priors expected utility and model uncertainty aversion

$$\min_{Q \in \mathcal{P}} \{E_Q[u(W)]\} \tag{1}$$

where

$$\mathcal{P} \equiv \{Q/E_Q[ln(dQ/dP)] < \eta\}$$

P is a reference probability distribution and η is a measure of uncertainty.

Preferences (1): multi-priors and uncertainty aversion

- Multi-priors expected utility and model uncertainty aversion

$$\min_{Q \in \mathcal{P}} \{E_Q[u(W)]\} \tag{1}$$

where

$$\mathcal{P} \equiv \{Q/E_Q[ln(dQ/dP)] < \eta\}$$

P is a reference probability distribution and η is a measure of uncertainty.

- Kogan and Wang (2002): normal distribution with common known variance,

$$\text{if } E_Q(r) = \mu + v, \text{ then } \mathcal{P} \equiv \{\mathcal{N}(\mu + v, \sigma^2)/v^2 \leq \phi_i^2\} \tag{2}$$

where μ is common to every agent.

Preferences (2): heterogenous model uncertainty

- Homogeneous CARA utility over final wealth, u(W)

$$u(W) = -e^{-\gamma W}$$

Preferences (2): heterogenous model uncertainty

- Homogeneous CARA utility over final wealth, u(W)

$$u(W) = -e^{-\gamma W}$$

- Heterogeneity in levels of uncertainty,

$$\phi_i \sim U(S), \quad S \equiv [\overline{\phi} - \delta, \overline{\phi} + \delta] \tag{3}$$

where $\delta \leq \overline{\phi}$ measures the dispersion of uncertainty among investors.

Portfolio choice

- Agent i problem,

$$\max_{D_i} \min_{Q \in \mathcal{P}} E_Q[u(W_{1i})]$$

$$s.t.$$

$$W_{1i} = xP + D_i(r - P)$$

Portfolio choice

- Agent i problem,

$$\max_{D_i} \min_{Q \in \mathcal{P}} E_Q[u(W_{1i})]$$

$$s.t.$$

$$W_{1i} = xP + D_i(r - P)$$

- Under normality and CARA utility,

$$\max_{D_i} \left\{ D_i(\mu - sgn(D_i)\phi_i - P) - \frac{\gamma\sigma^2}{2} D_i^2 \right\} \tag{4}$$

Portfolio choice

- Agent i problem,

$$\max_{D_i} \min_{Q \in \mathcal{P}} E_Q[u(W_{1i})]$$

$$s.t.$$

$$W_{1i} = xP + D_i(r - P)$$

- Under normality and CARA utility,

$$\max_{D_i} \left\{ D_i(\mu - sgn(D_i)\phi_i - P) - \frac{\gamma\sigma^2}{2} D_i^2 \right\} \quad (4)$$

- Solving for the optimal holding in stock,

$$D_i = \begin{cases} \frac{1}{\gamma\sigma^2}(\mu - \phi - P) & \text{if } \mu - P > \phi_i \\ 0 & \text{if } -\phi_i \leq \mu - P \leq \phi_i \\ \frac{1}{\gamma\sigma^2}(\mu + \phi - P) & \text{if } \mu - P < -\phi_i \end{cases} \quad (5)$$

Equilibrium Market Participation

- In equilibrium $\mu - P > 0$

Equilibrium Market Participation

- In equilibrium $\mu - P > 0$
- Full participation equilibrium

$$\begin{aligned} x &= \int_{\phi_i \in S} \frac{(\mu-\phi_i-P)}{\gamma\sigma^2}\frac{1}{2\delta}d\phi_i &= \frac{(\mu-\overline{\phi}-P)}{\gamma\sigma^2} \\ &\Rightarrow \mu - P &= \gamma\sigma^2 x + \overline{\phi} \end{aligned}$$

Full participation if $\gamma\sigma^2 x > \delta$

Equilibrium Market Participation

- In equilibrium $\mu - P > 0$
- Full participation equilibrium

$$\begin{aligned} x \quad = \quad & \int_{\phi_i \in S} \frac{(\mu - \phi_i - P)}{\gamma\sigma^2} \frac{1}{2\delta} d\phi_i & = & \quad \frac{(\mu - \overline{\phi} - P)}{\gamma\sigma^2} \\ \Rightarrow \quad & \mu - P & = & \quad \gamma\sigma^2 x + \overline{\phi} \end{aligned}$$

Full participation if $\gamma\sigma^2 x > \delta$

- Limited participation equilibrium

$$\begin{aligned} \mu - \phi^* - P &= 0 \\ x &= \int_{\phi_i < \phi^*, \phi_i \in S} \frac{(\mu - \phi_i - P)}{\gamma\sigma^2} \frac{1}{2\delta} d\phi_i \\ \Rightarrow \quad \phi^* &= \overline{\phi} - \delta + 2\sigma\sqrt{\gamma\delta x} = \mu - P \end{aligned}$$

Market Participation and Equity Premium

- Rewrite equity premium under limited participation as,

$$\mu - P = \frac{\gamma\sigma^2 x}{\alpha} + \phi_p \tag{6}$$

where

$$\alpha = \frac{1}{2\delta}(\phi^* - (\overline{\phi} - \delta)) = \sigma\sqrt{\frac{\gamma x}{\delta}} \tag{7}$$

is the proportion of investors holding stock and,

$$\phi_p = \frac{1}{2}(\phi^* + \overline{\phi} - \delta) = \overline{\phi} - \delta + \sigma\sqrt{\gamma\delta x} \tag{8}$$

is average level of uncertainty among market participants.

Market Participation and Equity Premium

- Rewrite equity premium under limited participation as,

$$\mu - P = \frac{\gamma\sigma^2 x}{\alpha} + \phi_p \tag{6}$$

where

$$\alpha = \frac{1}{2\delta}(\phi^* - (\overline{\phi} - \delta)) = \sigma\sqrt{\frac{\gamma x}{\delta}} \tag{7}$$

is the proportion of investors holding stock and,

$$\phi_p = \frac{1}{2}(\phi^* + \overline{\phi} - \delta) = \overline{\phi} - \delta + \sigma\sqrt{\gamma\delta x} \tag{8}$$

is average level of uncertainty among market participants.

Proposition

Suppose that $\gamma\sigma^2 x < \delta$. Then, there is limited market participation in equilibrium and,

$$\frac{\partial\alpha}{\partial\delta} < 0, \frac{\partial\phi_p}{\partial\delta} < 0, \frac{\partial(\mu - P)}{\partial\delta} < 0 \tag{9}$$

Uncertainty dispersion and Equity Premium

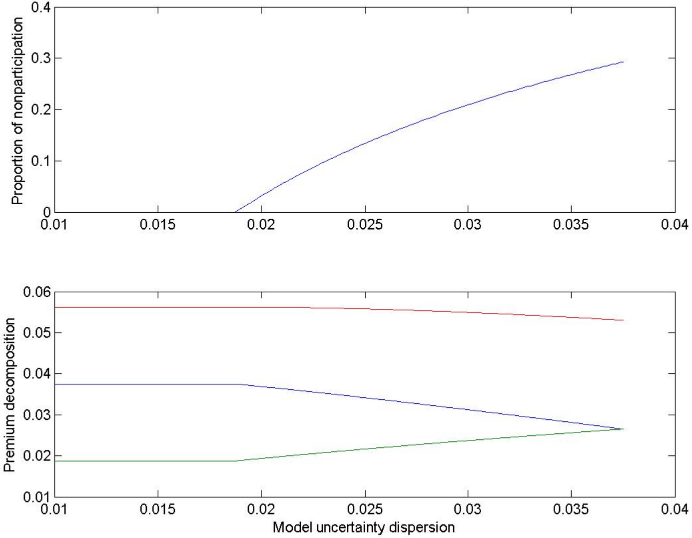

Figure 1. $x = 0.3, \sigma = 0.25, \mu = 1.06, \gamma = 1, \overline{\phi} = 0.0375$

Other results from the paper

Extensions:

1. Two stocks
   - Two stocks, and two factor portfolios
   - $r_M = r_1 + r_2$
   - Diversification discount: $P_M < P_1 + P_2$
   - With limited participation, price is determined by investors with less uncertainty for each firm. The price is lower for the bundle, because being less uncertain about one firm does not necessarily imply less uncertainty about the other.

Other results from the paper

Extensions:

1. Two stocks
    - Two stocks, and two factor portfolios
    - $r_M = r_1 + r_2$
    - Diversification discount: $P_M < P_1 + P_2$
    - With limited participation, price is determined by investors with less uncertainty for each firm. The price is lower for the bundle, because being less uncertain about one firm does not necessarily imply less uncertainty about the other.
2. Two periods
    - Introduce motive for hedging demand: non-tradable wealth correlated with asset return at t=1.
    - Introduce public signal to affect P_1
    - Without public signal, hedging demand dominates aversion to uncertainty: full participation.
    - With public signal, limited participation equilibrium exists.

Appendix 0: Stockholding puzzle

$$\underset{s_1,s_2}{Max} \quad U(c_0) + \beta E_0[U(c_1)]$$

s.t.

$$c_0 = s_1 + s_2 - Y_0$$

$$c_1 = s_1 R_1 + s_2 R_2 + Y_1$$

$\Rightarrow$ FOCs:

$$U'(c_0) = \beta E_0 \left[R_1 U'(c_1) \right]$$

$$U'(c_0) = \beta E_0 \left[R_2 U'(c_1) \right]$$

$\Rightarrow$

$$0 = E_0 \left[(R_1 - R_2) U'(c_1) \right]$$

Appendix 00: Stockholding puzzle

Applying Cov and BC, assuming R_2 risk free,

$$0 = (E_0[R_1] - R_2)E_0[U'(c_1)] + Cov_0[R_1 U'(s_1 R_1 + s_2 R_2 + Y_1)]$$

if $s_1 = 0$ is solution,

$$0 = (E_0[R_1] - R_2)E_0[U'(s_2 R_2 + Y_1)] + Cov_0[R_1 U'(s_2 R_2 + Y_1)]$$

assuming $Cov(Y_1, R_1) = 0$,

$$0 = (E_0[R_1] - R_2)E_0[U'(s_2 R_2 + Y_1)]$$

With positive E.P. ($E_0[R_1] - R_2 > 0$) and no bliss point ($E_0[U'(c_1)] > 0$) this is a contradiction.

Appendix 1: proof Proposition 1

Proposition

Suppose that $\gamma\sigma^2 x < \delta$. Then, there is limited market participation in equilibrium and,

$$\frac{\partial \alpha}{\partial \delta} < 0, \frac{\partial \phi_p}{\partial \delta} < 0, \frac{\partial(\mu - P)}{\partial \delta} < 0 \tag{10}$$

$$\frac{\partial(\mu-P)}{\partial\delta} = \frac{\partial \frac{\gamma\sigma^2 x}{\alpha}}{\partial\delta} + \frac{\partial\phi_p}{\partial\delta}$$

where

$$\frac{\partial \frac{\gamma\sigma^2 x}{\alpha}}{\partial\delta} = -\delta\frac{\partial\alpha}{\partial\delta} = \delta^{-1/2}\sigma\sqrt{\gamma x}\frac{1}{2}$$

$$\frac{\partial\phi_p}{\partial\delta} = \delta^{-1/2}\sigma\sqrt{\gamma x}\frac{1}{2} - 1$$

$$\Rightarrow$$

$$\frac{\partial(\mu-P)}{\partial\delta} = \delta^{-1/2}\sigma\sqrt{\gamma x} - 1 \quad < 0, \quad \text{if } \gamma\sigma^2 x < \delta$$

Appendix 2: Model uncertainty - Matlab

- Three possible values: $w_1 = 0$, $w_2 = 10$, $w_3 = 20$
- $\sigma^2 = 25$
- Different probability distributions,

π_1	π_2	π_3	μ
0.1000	0.7472	0.1528	10.5279
0.2000	0.7325	0.0675	8.6754
0.3000	0.6746	0.0254	7.2540
0.4000	0.5944	0.0056	6.0557
0.5000	0.5000	0	5.0000
0.6000	0.3954	0.0046	4.0455
0.7000	0.2832	0.0168	3.1678
0.8000	0.1649	0.0351	2.3509
0.9000	0.0416	0.0584	1.5836

Appendix 3: Some evidence

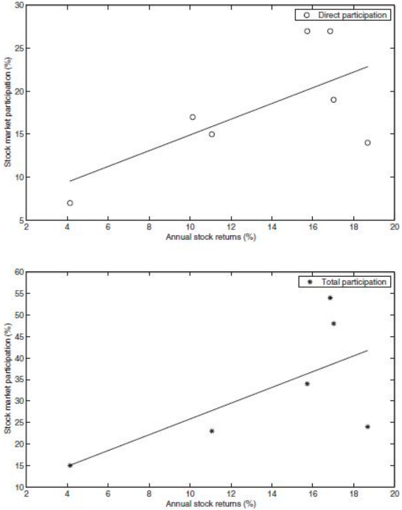

Figure 2
Stock market participation and total annual stock returns for France, Germany, Italy, the Netherlands, Sweden, the United Kingdom, and the United States

Two Stocks

- Two stocks, and two factor portfolios

$$\begin{aligned} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} &= \beta \begin{pmatrix} f_A \\ f_B \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \\ r_M &= r_1 + r_2 \end{aligned}$$

- Diversification discount: $P_M < P_1 + P_2$
- With limited participation, a firm's pre-merger price is determined by investors with less uncertainty for that firm. After the merger investors have to buy both firms as a bundle, but being less uncertain about one firm does not necessarily imply less uncertainty about the other, so they offer a lower price for the combined firm.

Appendix 5: Dynamic Model

Two periods

- Introduce motive for hedging demand: non-tradable wealth correlated with asset return at t=1.
- Introduce public signal to affect P_1
- Without public signal, hedging demand dominates aversion to uncertainty: full participation.
- With public signal, additional uncertainty about P_1. Under some parameters, limited participation.

Appendix 6: Static Ellsberg(1961)'s Experiment

- Challenge Savage(1954)'s expected utility with subjective probabilities framework.
- 30 balls in an urn, 10 are yellow, and 20 are either blue or green.
- Two choice problems:

		Yellow	Blue	Green
Choice I	f_1	4	0	0
	f_2	0	4	0
Choice II	f_3	4	0	4
	f_4	0	4	4

- Picking f_1, f_4 denotes *ambiguity* (or *uncertainty*) *aversion*, picking f_2, f_3 denotes *love for ambiguity*.
- Both picks violate Savage's axioms and denote an attitude towards uncertainty.