On Efficient Distribution with Private Information
by Atkeson and Lucas - RES 1992

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October 2010
The model

- \( t = 0, 1, 2, \ldots \)
- Single, non storable consumption good, \( y_t = y \)
- Continuum of consumers
  - Preferences
    \[
    E \left( \sum_{t=0}^{\infty} (1 - \beta) \beta^t V(c_t) \theta_t \right)
    \]
    \( V' > 0, \ V'' < 0, \ \theta_t \in \Theta = \{\theta_1, \ldots, \theta_n\} \)
    \[
    \Pr (\theta_t = \theta_i) = \mu (\theta_i) \text{ for all } i = 1, \ldots n \text{ and all } t
    \]
  - Endowment: \( w \in D \subset \mathbb{R} \)
    \[
    \Pr (w \in A) = \psi (A) \text{ for all } A \subseteq D
    \]
- \( \theta_t \) is private information.
Minimize constant $y$ to attain a certain distribution $\psi$ given the information available.
Reporting Strategies

- **Reporting strategy:**
  
  \[ z = \{ z_t (\theta^t) \}_{t=0}^{\infty} \]

- **Truthful** reporting strategy: \( z^* = \{ z^*_t (\theta^t) \}_{t=0}^{\infty} \)

  \[ z^*_t (\theta^t) = \theta^t \quad \forall t, \; \forall \theta^t \in \Theta^{t+1}. \]
Reporting Strategies

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  \[ z = \{ z_t (\theta^t) \}_{t=0}^\infty \]

- **Truthful** reporting strategy: \[ z^* = \{ z^*_t (\theta^t) \}_{t=0}^\infty \]
  \[ z^*_t (\theta^t) = \theta_t \quad \forall \ t, \ \forall \theta^t \in \Theta^{t+1}. \]

- Reporting history: \[ z^t = \{ z_0 (\theta_0), ..., z_t (\theta_t) \} \]
Plan

- **Plan**: \( u = \{ u_t (w, z^t) \}_{t=0}^{\infty} \) such that

\[
\lim_{t \to \infty} \beta^t \sum_{s=0}^{\infty} \beta^s u_{t+s} (w, z^{t+s}) \theta_{t+s} = 0
\]

where \( u_t (w, z^t) = V (c_t (w, z^t)) \).
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  \]
  where \( u_t (w, z^t) = V (c_t (w, z^t)) \).

- Expected discounted utility
  \[
  U (w, u, z) \equiv E \left( \sum_{t=0}^{\infty} (1 - \beta) \beta^t u (w, z^t) \theta_t \right)
  \]
A plan $u$ is an **allocation** if it satisfies

(i) Truth Telling:

$$U(w, u, z^*) \geq U(w, u, z) \quad \text{for all } z \in Z \text{ and all } w \in D$$

(ii) Promise keeping:

$$U(w, u, z^*) = w \quad \text{for all } w$$
Planner’s Problem

\[
\min_{u \in A} y \\
\text{such that} \\
\int_{D \times \Theta} C(u_t(w, \theta^t)) \, d\mu(\theta_t) \, d\psi(w) \leq y \text{ for all } t
\]

where \( C(u_t(w, z^t)) \equiv c_t(w, z^t) \).
Allocation rule

A sequence $\sigma = \{f_t, g_t\}_{t=0}^{\infty}$ is an allocation rule if it satisfies

(i) Promise keeping: $\forall w_t, \forall t$

$$\int_\Theta [(1 - \beta) f_t (w_t, \theta_t) \theta_t + \beta g_t (w_t, \theta_t)] \, d\mu = w_t$$

(ii) Truth telling: $\forall z_t \in \Theta, \forall w_t \in D, \forall t$

$$[(1 - \beta) f_t (w_t, \theta_t) \theta_t + \beta g_t (w_t, \theta_t)] \, d\mu \geq [(1 - \beta) f_t (w_t, z_t) \theta_t + \beta g_t (w_t, z_t)] \, d\mu$$

(iii) Boundedness: $\lim_{t \to \infty} \beta^t g_t (w_t (w_0, z^{*t-1}), z^*_t) = 0$. 
Planner’s Problem

\[
\min_{\sigma \in R} y \\
\text{such that for all } t
\]

\[
\int_{D \times \Theta} C(f_t(w_t, \theta)) \, d\mu(\theta) \, d\psi_t(w) \leq y
\]

where

\[
\psi_{t+1}(w) = S_g(\psi_t)(w) \equiv \int_{D \times \Theta} I\{w_t : g_t(w_t, \theta) = w\} \, d\mu \, d\psi_t.
\]
Equivalence

- \( R(y, \psi) \): be the set of allocations /allocation rules that attain \( \psi \) with resources \( y \).
- Then,

\[
\begin{align*}
  u \in R(y, \psi) & \rightarrow \sigma \in R(y, \psi) \\
  \sigma \in R(y, \psi) & \rightarrow u' = \{ f_t(w_t(w_0, z^{t-1}), z_t) \} \in R(y, \psi)
\end{align*}
\]

where

\[
g_t(w_t, z_t) = w_{t+1}
\]
Bellman Equation

\( \phi^* (\psi) \): minimum cost of attaining a distribution \( \psi \).

\[
\phi (\psi) = \inf_{f,g} \max \left\{ \int_{D \times \Theta} C (f(w, \theta)) \, d\mu (\theta) \, d\psi (w), \phi^* (S_g \psi) \right\}
\]
Problem T

Lemma 4.1 \( \varphi^* \) is a fixed point of \( T \) where

\[
(T \varphi)(\psi) = \inf_{f, g \in B} \max \left\{ \int_{D \times \Theta} C(f(w, \theta)) \, d\mu(\theta) \, d\psi(w), \varphi(S_g \psi) \right\}
\]
Problem T

Lemma 4.2 If there are functions $\varphi_a$, $\varphi_c$ and $\varphi$ such that $\forall \psi$

- $\varphi_c (\psi) < \varphi^* (\psi) < \varphi_a (\psi)$
- $\lim_{n \to \infty} T^n \varphi_a (\psi) = \lim_{n \to \infty} T^n \varphi_c (\psi) = \varphi (\psi)$

Then, $\varphi = \varphi^*$. 
Problem T

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Then, $\varphi = \varphi^*$.

- Candidates:
  - Autarky: $\varphi_a(\psi) = \int_D C(w) \, d\psi$
  - Complete insurance: $\varphi_c(\psi) = \int_{D \times \Theta} C(u_c(w, \theta)) \, d\psi \, d\varphi$
Example: log utility

- $V(x) = \log(x)$, $C(u) = \exp(u)$

- Bounding functions
  - $\varphi_a(\psi) = \int_D \exp(w) \, d\psi$
  - $\varphi_c(\psi) = \alpha \int_D \exp(w) \, d\psi$ where $\alpha = \exp\{ -E[\theta \log \theta] \}$

- The degree of inequality grows without bound when resources are efficiently allocated.
• **Lemma 3.1** \( \forall \psi \in M, \text{ if } u \in A \text{ attains } \psi \text{ with resources } y, \exists \sigma \in R \text{ that attains } \psi \text{ with resources } y. \)

• **Lemma 3.2** \( \forall \psi \in M, \text{ if } \sigma \in R \text{ attains } \psi \text{ with resources } y \text{ and } u = \text{ is the utility plan generated by } \sigma, \text{ then } u \in A \text{ and attains } \psi \text{ with resources } y. \)