Leaning against the wind
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Model

- $t \in [0, \infty)$
- 1 good "cash"
- 1 asset in fixed supply $s$, each unit valued $\theta^i_t$ by agent $i$
- 2 types of agents:
  - competitive marketmakers (MM): $\theta_t = 0 \ \forall t \geq 0$, access to return $r$, a (0) cash.
  - investors: $\theta_t \in \{1, 1-\delta\}$, $\delta \in (0, 1)$,
    \[
    \Pr(\theta_{t+1} = 1-\delta|\theta_t = 1) = \gamma_d \\
    \Pr(\theta_{t+1} = 1|\theta_t = 1-\delta) = \gamma_u
    \]
- $\mu_\sigma(t)$: measure of investors of type $\sigma \in \{hn, ho, ln, lo\}$
Initial conditions

Table 1: Initial conditions.

<table>
<thead>
<tr>
<th>$\mu_{lo}(0)$</th>
<th>$\mu_{hn}(0)$</th>
<th>$\mu_{ln}(0)$</th>
<th>$\mu_{ho}(0)$</th>
<th>$I(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>0</td>
<td>$1 - s$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Asset Market

- Investors only trade with a marketmaker.
- Meeting process: Poisson with rate $\rho$
- Asset flows: $u_I(t)$, $u_H(t)$.

$$\begin{align*}
-\rho \mu_{ln}(t) & \leq u_I(t) \leq \rho \mu_{lo}(t) \\
-\rho \mu_{ho}(t) & \leq u_H(t) \leq \rho \mu_{hn}(t)
\end{align*}$$

- MM hold $I(t)$ units of asset at time $t$.

$$\mu_{lo}(t) + \mu_{ho}(t) + I(t) = s \text{ for all } t \geq 0$$

- Inventory:

$$\begin{align*}
\dot{I}(t) &= u_I(t) - u_H(t) \\
I(t) &\geq 0
\end{align*}$$
Feasibility

Definition

A feasible allocation is some distribution \( \mu(t) = \{\mu_\sigma(t)\}_\sigma \) of types, an inventory holding \( l(t) \), and asset flows \( u(t) = (u_l(t), u_h(t)) \) such that (1), (2), and the initial conditions hold and

\[
\begin{align*}
\dot{\mu}_{lo}(t) &= -u_l(t) - \gamma_u \mu_{lo}(t) + \gamma_d \mu_{ho}(t) \\
\dot{\mu}_{hn}(t) &= -u_h(t) - \gamma_d \mu_{hn}(t) + \gamma_u \mu_{ln}(t) \\
\dot{\mu}_{ln}(t) &= u_l(t) - \gamma_u \mu_{ln}(t) + \gamma_d \mu_{hn}(t) \\
\dot{\mu}_{ho}(t) &= u_h(t) - \gamma_d \mu_{ho}(t) + \gamma_u \mu_{lo}(t).
\end{align*}
\]
Socially optimal allocation

Definition

A **socially optimal allocation** is a feasible allocation that maximizes

$$
\int_0^\infty e^{-rt} \left( \mu_{ho}(t) + (1 - \delta) \mu_{lo}(t) \right) dt
$$
Buffer Allocation

Definition

A **buffer allocation** is a feasible allocation defined by two times $(t_1, t_2)$, $t_1 \leq t_2$ called breaking times such that:

\[
\begin{align*}
    u_l(t) &= \rho \mu_{hn}(t) \quad \text{and} \quad u_h(t) = \rho \mu_{hn}(t) \quad \text{for} \quad t \in [0, t_1] \\
    u_l(t) &= \rho \mu_{lo}(t) \quad \text{and} \quad u_h(t) = \rho \mu_{hn}(t) \quad \text{for} \quad t \in (t_1, t_2) \\
    u_l(t) &= \rho \mu_{lo}(t) \quad \text{and} \quad u_h(t) = \rho \mu_{lo}(t) \quad \text{for} \quad t \in [t_2, \infty)
\end{align*}
\]
Figure: Buffer allocation
Constrained Efficiency

Theorem

The socially optimal allocation \((\mu^*(t), l^*(t), u^*(t), t \geq 0)\) is a buffer allocation with breaking times \((t_1^*, t_2^*)\) where:

\[
\begin{align*}
t_1^* &< t_2^* \\
t_1^* &\geq 0 \\
t_2^* &< \infty
\end{align*}
\]
Marketmaker’s Problem

Choose \( \{a(t), l(t), u(t), c(t), t \geq 0\} \) to maximize

\[
\int_0^\infty e^{-rt} c(t) \, dt
\]

s.t.

\[
\begin{align*}
\dot{a}(t) &= ra(t) + p(t) (u_l(t) - u_h(t)) - c(t) \\
\dot{l}(t) &= u_l(t) - u_h(t) \\
l(t) &\geq 0 \\
a(t) &\geq 0
\end{align*}
\]
Investor’s Problem

Equilibrium strategy:

- $ho$ and $ln$ never trade
- $lo$ are sellers
- $hn$ are buyers

Reservation values given this strategy:

for the seller : $\Delta V_l(t)$
for the buyer : $\Delta V_h(t)$
**Theorem**

There exists $\bar{a}_0^* \in \mathbb{R}_+$ such that $\forall a(0) > \bar{a}_0^*$, $\exists$ a competitive equilibrium whose allocation is optimal.
Figure: Competitive equilibrium for $a(0) > \bar{a}_0^*$
Borrowing constrained Marketmakers

Theorem

There exists $a_0^* \leq \bar{a}_0^*$ such that:

(i) If $a(0) \in [0, a_0^*)$ there exists an equilibrium whose allocation is a buffer allocation and the liquidity provision is less than the optimal.

(ii) If $a(0) \in [a_0^*, \bar{a}_0^*)$ there exists an equilibrium whose allocation is a buffer allocation and the liquidity provision is the optimal.

If $t_1^* > 0$ then $\underline{a}_0^* = \bar{a}_0^*$. In all of the above, the equilibrium price is strictly positive and has a jump at the maximum inventory position.
Figure: Equilibrium Price
Figure: Equilibrium Allocations
Policy Implication

- Bank loan subsidization
- Not always good for MM to provide liquidity when selling pressure is highest.