Leaning against the wind

Pierre Olivier Weill 2007

Cecilia Parlatore Siritto

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Model

- $t \in [0, \infty)$
- 1 good "cash"
- 1 asset in fixed supply s, each unit valued θ_t^i by agent i
- 2 types of agents:
  - competitive marketmakers (MM): $\theta_t = 0\ \forall t \geq 0$, access to return r, $a(0)$ cash.
  - investors: $\theta_t \in \{1, 1-\delta\}$, $\delta \in (0,1)$,

$$\begin{aligned} \Pr\left(\theta_{t+1} = 1-\delta | \theta_t = 1\right) &= \gamma_d \\ \Pr\left(\theta_{t+1} = 1 | \theta_t = 1-\delta\right) &= \gamma_u \end{aligned}$$

- $\mu_\sigma(t)$: measure of investors of type $\sigma \in \{hn, ho, ln, lo\}$

Initial conditions

Table 1: Initial conditions.

$\mu_{\ell o}(0)$	$\mu_{hn}(0)$	$\mu_{\ell n}(0)$	$\mu_{ho}(0)$	$I(0)$
s	0	$1-s$	0	0

Asset Market

- Investors only trade with a marketmaker.
- Meeting process: Poisson with rate ρ
- Asset flows: $u_l(t)$, $u_h(t)$.

$$\begin{aligned} -\rho\mu_{ln}(t) &\leq u_l(t) \leq \rho\mu_{lo}(t) \\ -\rho\mu_{ho}(t) &\leq u_h(t) \leq \rho\mu_{hn}(t) \end{aligned} \qquad (1)$$

- MM hold $I(t)$ units of asset at time t.

$$\mu_{lo}(t) + \mu_{ho}(t) + I(t) = s \text{ for all } t \geq 0$$

- Inventory:

$$\begin{aligned} \dot{I}(t) &= u_l(t) - u_h(t) \\ I(t) &\geq 0 \end{aligned} \qquad (2)$$

Feasibility

Definition

A **feasible allocation** is some distribution $\mu(t) = \{\mu_\sigma(t)\}_\sigma$ of types, an inventory holding $I(t)$, and asset flows $u(t) = (u_l(t), u_h(t))$ such that $(1), (2)$, and the initial conditions hold and

$$
\begin{aligned}
\dot{\mu}_{lo}(t) &= -u_l(t) - \gamma_u \mu_{lo}(t) + \gamma_d \mu_{ho}(t) \\
\dot{\mu}_{hn}(t) &= -u_h(t) - \gamma_d \mu_{hn}(t) + \gamma_u \mu_{ln}(t) \\
\dot{\mu}_{ln}(t) &= u_l(t) - \gamma_u \mu_{ln}(t) + \gamma_d \mu_{hn}(t) \\
\dot{\mu}_{ho}(t) &= u_h(t) - \gamma_d \mu_{ho}(t) + \gamma_u \mu_{lo}(t).
\end{aligned}
$$

Socially optimal allocation

Definition

A **socially optimal allocation** is a feasible allocation that maximizes

$$\int_0^\infty e^{-rt} \left(\mu_{ho}(t) + (1-\delta)\,\mu_{lo}(t)\right) dt$$

Buffer Allocation

Definition

A **buffer allocation** is a feasible allocation defined by two times (t_1, t_2), $t_1 \leq t_2$ called breaking times such that:

$$
\begin{aligned}
u_l(t) &= \rho\mu_{hn}(t) \text{ and } u_h(t) = \rho\mu_{hn}(t) \text{ for } t \in [0, t_1] \\
u_l(t) &= \rho\mu_{lo}(t) \text{ and } u_h(t) = \rho\mu_{hn}(t) \text{ for } t \in (t_1, t_2) \\
u_l(t) &= \rho\mu_{lo}(t) \text{ and } u_h(t) = \rho\mu_{lo}(t) \text{ for } t \in [t_2, \infty)
\end{aligned}
$$

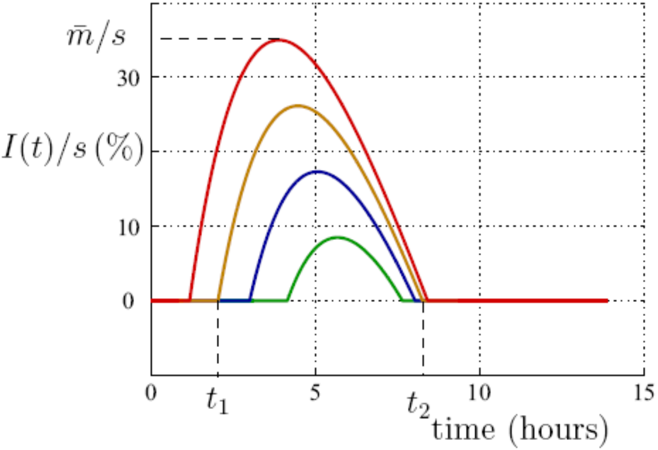

Figure: Buffer allocation

Constrained Efficiency

Theorem

The socially optimal allocation $(\mu^*(t), I^*(t), u^*(t), t \geq 0)$ *is a buffer allocation with breaking times* (t_1^*, t_2^*) *where:*

$$
\begin{aligned}
t_1^* &< t_2^* \\
t_1^* &\geq 0 \\
t_2^* &< \infty
\end{aligned}
$$

Marketmaker's Problem

Choose $\{a(t), I(t), u(t), c(t), t \geq 0\}$ to maximize

$$\int_0^\infty e^{-rt} c(t)\, dt$$

s.t.

$$\begin{aligned} \dot{a}(t) &= ra(t) + p(t)(u_l(t) - u_h(t)) - c(t) \\ \dot{I}(t) &= u_l(t) - u_h(t) \\ I(t) &\geq 0 \\ a(t) &\geq 0 \end{aligned}$$

Investor's Problem

Equilibrium strategy:

- *ho* and *ln* never trade
- *lo* are sellers
- *hn* are buyers

Reservation values given this strategy:

$$\begin{aligned} \text{for the seller} &: \quad \Delta V_l(t) \\ \text{for the buyer} &: \quad \Delta V_h(t) \end{aligned}$$

Efficiency

Theorem

There exists $\bar{a}_0^* \in \mathbb{R}_+$ *such that* $\forall a(0) > \bar{a}_0^*$, $\exists$ *a competitive equilibrium whose allocation is optimal.*

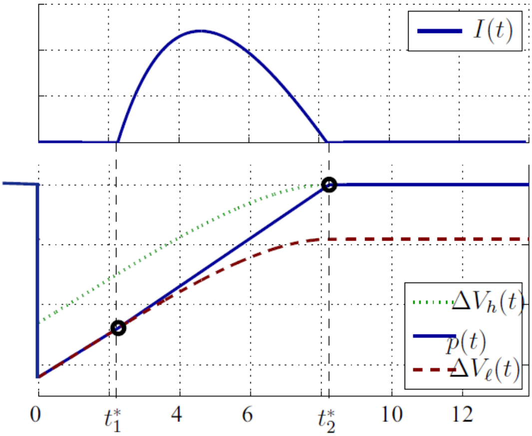

Figure: Competitive equilibrium for $a(0) > \bar{a}_0^*$

Borrowing constrained Marketmakers

Theorem

There exists $\underline{a}_0^* \leq \bar{a}_0^*$ *such that:*
(i) *If* $a(0) \in [0, \underline{a}_0^*)$ *there exists an equilibrium whose allocation is a buffer allocation and the liquidity provision is less than the optimal.*
(ii) *If* $a(0) \in [\underline{a}_0^*, \bar{a}_0^*)$ *there exists an equilibrium whose allocation is a buffer allocation and the liquidity provision is the optimal.*
If $t_1^* > 0$ *then* $\underline{a}_0^* = \bar{a}_0^*$. *In all of the above, the equilibrium price is strictly positive and has a jump at the maximum inventory position.*

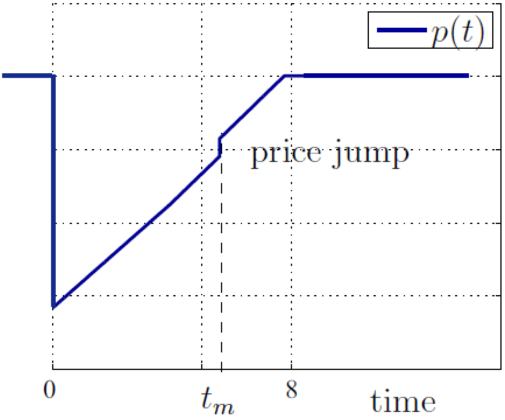

Figure: Equilibrium Price

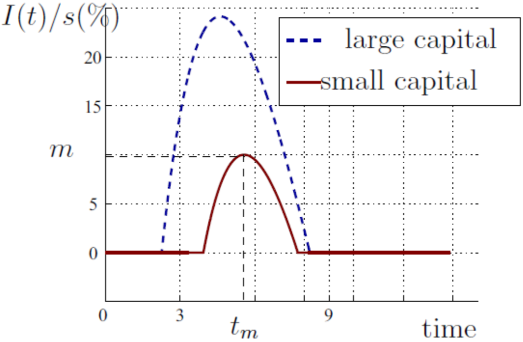

Figure: Equilibrium Allocations

Policy Implication

- Bank loan subsidization
- Not always good for MM to provide liquidity when selling pressure is highest.