Default and Aggregate Income
by Rampini - JET 2005

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The Model

- $t = 0, 1$
- 2 goods: consumption and leisure
- Continuum of ex-ante identical agents
  - Preferences
    $$U(c, h) = \log(c) + v(h)$$
    $v' > 0$, $v'' < 0$, $\lim_{h \to 0} v(h) = \infty$, $v(H) = 0$.
  - Endowment: (e, H)
    $$e = \omega + \theta$$
    Aggregate, observable: $\omega \sim F(\omega)$, $\omega \in \Omega = [\omega_0, \omega_1]$

    Idiosyncratic, private information: $\theta \sim \mathbb{U}(\theta)$, $\theta \in \Theta = [0, 1]$.

Contracts

- Contract: $\left(r\left(\hat{\theta},\omega\right),h\left(\hat{\theta},\omega\right)\right)$
- Indirect utility

$$U\left(\hat{\theta},\theta,\omega\right)=\log\left(\omega+\theta-r\left(\hat{\theta},\omega\right)\right)+v\left(h\left(\hat{\theta},\omega\right)\right).$$

- Penalty:

$$t\left(\hat{\theta},\omega\right)=v\left(H\right)-v\left(h\left(\hat{\theta},\omega\right)\right)=-v\left(h\left(\hat{\theta},\omega\right)\right)$$

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Optimal Risk Sharing

$$\max_{\{r(\theta,\omega),t(\theta,\omega)\}} \int_{\Omega} \int_{\Theta} U(\theta, \theta, \omega)\, d\theta dF(\omega)$$

subject to

$$\begin{aligned} \theta &\in \arg\max_{\hat{\theta}\in\Theta} U\left(\hat{\theta}, \theta, \omega\right), \ \forall \omega \in \Omega \ \forall \theta \in \Theta \\ \int_{\Theta} r(\theta, \omega) &\geq 0 \ \forall \omega \in \Omega \\ t(\theta, \omega) &\geq 0, \ \forall \theta \in \Theta, \ \forall \omega \in \Omega \end{aligned}$$

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- Can solve pointwise for $\omega\in\Omega$.

Optimal repayment schedule

- $\theta \in \left[0, \bar{\theta}(\omega)\right)$, $r(\theta, \omega) = r\left(\bar{\theta}, \omega\right)$ and $t(\theta, \omega) > 0$
- $\theta \in \left[\bar{\theta}, \theta^*(\omega)\right]$,

$$r(\theta, \omega) = \omega + \theta - u''^{-1}\left(-\frac{a}{\theta - \theta_0}\right)$$

  where $u = \log$, $a > 0$, and $t(\theta, \omega) > 0$.
- $\theta \in (\theta^*(\omega), 1]$, $r(\theta, \omega) = r(\theta^*, \omega)$ and $t(\theta, \omega) = 0$

Default

- Amount of people who default: $\theta^* (\omega)$
- Total amount defaulted:

$$
\begin{aligned}
& \int_0^{\theta^*(\omega)} \left[r\left(\theta^*, \omega\right) - r\left(\theta, \omega\right) \right] d\theta \\
= \; & \left[r\left(\theta^*\left(\omega\right), \omega\right) - r\left(\bar{\theta}\left(\omega\right), \omega\right) \right] \bar{\theta}\left(\omega\right) \\
& + \int_{\bar{\theta}(\omega)}^{\theta^*(\omega)} \left[r\left(\theta^*, \omega\right) - r\left(\theta, \omega\right) \right] d\theta
\end{aligned}
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& + \int_{\bar{\theta}(\omega)}^{\theta^*(\omega)} \left[r(\theta^*, \omega) - r(\theta, \omega) \right] d\theta
\end{aligned}
$$

- Maximum default

$$
\left[r(\theta^*(\omega), \omega) - r\left(\bar{\theta}(\omega), \omega\right) \right]
$$

Proposition Under the assumptions made:

- $\theta^* (\omega)$ is decreasing in ω
- $\left[r (\theta^*, \omega) - r \left(\bar{\theta} (\omega) , \omega\right)\right]$ is decreasing in ω

Changes in aggregate Income

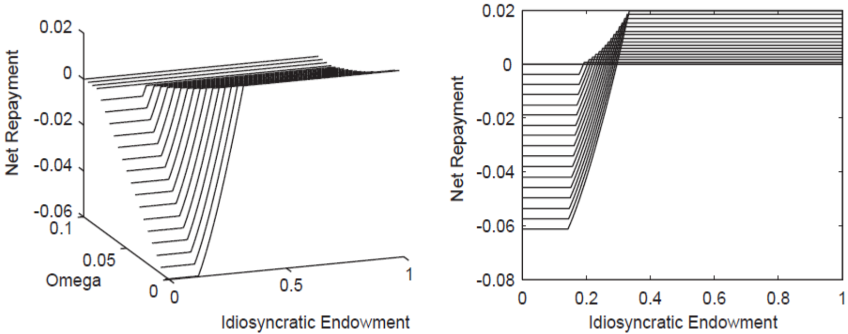
$\exists$ unique ω^0 such that $\theta^*\left(\omega^0\right) = \bar{\theta}\left(\omega^0\right) \in (0,1)$.

- For $\omega \geq \omega^0$ there is no default:

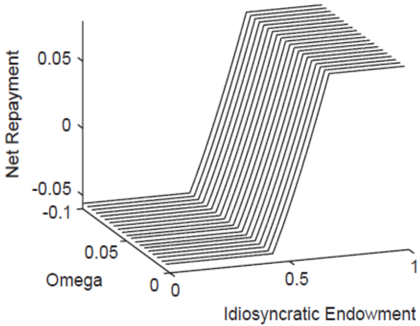
$$r(\theta,\omega) = t(\theta,\omega) = 0 \ \forall \theta \in \Theta$$

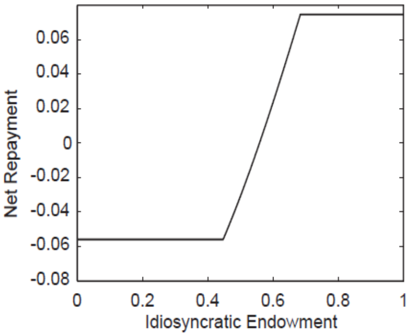
- For $\omega < \omega^0$ $\theta^*(\omega) > \theta^*\left(\omega^0\right)$, $\theta^*(\omega) >> 0$.

Net Repayment Schedule - Log utility



Net Repayment Schedule CARA





Implementability

$r:\Theta\times\Omega\rightarrow\mathbb{R}$ is implementable if $\exists t:\Theta\times\Omega\rightarrow\mathbb{R}$ such that $\left(r\left(\theta\right),t\left(\theta\right)\right)$ satisfies

$$\theta\in\arg\max_{\hat{\theta}\in\Theta}U\left(\hat{\theta},\theta,\omega\right)=\arg\max_{\hat{\theta}\in\Theta}\log\left(\omega+\theta-r\left(\hat{\theta},\omega\right)\right)-t\left(\hat{\theta},\omega\right)$$

for all $\theta\in\Theta$.

- Necessary condition: $r'\left(\theta\right)>0$.