Long-Term Risk: An Operator Approach

LP Hansen and JA Scheinkman (2009)

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Motivation

Goals of this paper

1. Introduce operator methods in analysis of dynamic systems
2. Formally describe the mathematical framework to apply operator methods
3. Discuss a few applications in asset pricing
   - Term Structure of Risk
   - Why should we care about it?
Continuous time nonlinear Markov environment

1. Introduce Operator semigroups

\[ \mathbb{M}_t \psi(x) = E[M_t \psi(X_t) | X_0 = x] \]

2. Obtain Multiplicative Factorization

\[ M_t = \exp(\rho t) \hat{M}_t \frac{\phi(X_0)}{X_t} \]

3. Characterize \((\rho, \phi)\) as a solution to an eigenvalue - eigenfunction problem

\[ \mathbb{M}_t \phi(x) = \exp(\rho t) \phi(x) \]

4. Establish existence and uniqueness of this decomposition
Underlying Markov process

Given a probability space \( \{ \Omega, \mathcal{F}, P \} \),

Let \( \{ X_t : t \geq 0 \} \) be strong Markov, RCLL with the natural filtration.

Examples

1. \( D_0 \) is finite : Finite-State Markov Chain with intensity matrix \( \mathcal{U} \)
2. \( D_0 \) is infinite : Semimartingales with continuous and pure jump components

In particular \( X_t \) is an Ito process with following local evolution:

\[
\begin{align*}
    dX_t^f &= \xi_f ( \bar{x}_f - X_t^f ) dt + \sqrt{X_t^f} \sigma_f dB_t^f \\
    dX_o^o &= \xi_o ( \bar{x}_o - X_t^o ) dt + \sigma_o dB_t^o
\end{align*}
\]

Definition

A functional $M$ is a real-values process $\{M_t : t \geq 0\}$ that is adapted and has RCLL paths.

Definition

A functional $M$ is multiplicative if,

1. $M_0 = 1$
2. $M_{t+u} = M_u(\Theta_t)M_t \quad \forall t, u$

Here $\Theta_\tau$ is the “shift-operator”

$$\Theta_\tau(X) = Y \quad Y \equiv \{Y_t = X_{t+\tau} : t \geq 0\}$$
**Definition**

A functional $A$ is **additive** if,

1. $A_0 = 0$
2. $A_{t+u} = A_u(\Theta_t) + A_t \quad \forall t, u$

**Definition**

A **Valuation functional** $\{V_t : t \geq 0\}$ is a multiplicative functional such that $\{V_t S_t : t \geq 0\}$ is a martingale for some strictly positive multiplicative functional $S_t$.
Let $\langle L, \| . \| \rangle$ be a Banach space and $\{T_t : t \geq 0\}$ be a family of operators on $L$.

**Definition**

A family of linear operators $\{T_t : t \geq 0\}$ is a one-parameter semigroup if,

1. $T_0 = I$
2. $T_{t+s} = T_t T_s \quad \forall t, s$
Additive, multiplicative functionals are closed under respective binary operations.

If $A$ is additive, then $M = \exp(A)$ is multiplicative.

Parametrization: Additive functional $A_t = \langle \beta^A, \gamma^A \rangle$

$$A_t = \int_0^t \beta^a(X_u)du + \int_0^t \gamma^a(X_u)dB_u$$

Further whenever $M = \exp(A)$, for simplicity, we would also parameterize $M$ with $\langle \beta^A, \gamma^A \rangle$.

If $M$ is a multiplicative functional such that for all $\psi \in L$, $E[M_t\psi(X_t)|X_0 = x] \in L$, then $M \equiv \{M_t\}$ is a one-parameter semigroup,

$$M_t = E[M_t\psi(X_t)|X_0 = x]$$
1. Stochastic Discount Factors: To the previously described “Feller-OU” setup add the following two things:

- Log consumption: \( dc_t = X_t^o \, dt + \sqrt{X_t^f} \, \psi_f \, dB_t^f + \psi_o \, dB_t^o \)
- Preferences: \( E \int_0^\infty \exp(-bt) \frac{C_t^{1-a}}{1-a} \, dt \)

The implied stochastic discount factor is \( S_t = \exp(A_t^s) \) where

\[
dA_t^s = (-aX_t^0 - b) \, dt - a\sqrt{(X_t^f)} \, \psi_f \, dB_t^f - a\psi_o \, dB_t^o
\]

2. Growth Functionals: \( D_t = G_t \psi(X_t)D_0 \) where \( G_t \) is a multiplicative functional

3. Semigroups: Conditional expectation operators
A Borel function $\psi$ belongs to the domain of the extended generator $\mathbb{A}_M$ of the multiplicative functional $M$ if there exists a Borel function $\chi$ such that $N_t = M_t \psi(X_t) - \psi(X_0) - \int_0^t M_s \chi(X_s) \, ds$ is a local martingale with respect to the filtration $\{\mathcal{F}_t : t \geq 0\}$.

In this case, the extended generator assigns the function $\psi$ to $\chi$ and we write $\mathbb{A}_M \psi = \chi$.

Formalizes the concept of the “expected time derivative”.

Generators-Examples

Let $X_t = \langle \xi, \Gamma \rangle$

- $M_t = 1$

$$A_I \phi(x) = \xi(x) \cdot \frac{\partial \phi(x)}{\partial x} + \frac{1}{2} \text{trace} \left( \Gamma(x) \Gamma(x)' \frac{\partial^2 \phi(x)}{\partial x \partial x'} \right)$$

- For general $M_t = \langle \beta, \gamma \rangle$

$$A_M \phi(x) = \left( \beta(x) + \frac{|\gamma(x)|^2}{2} \right) \phi(x) + \left( \xi(x) + \Gamma(x) \gamma(x) \right) \cdot \frac{\partial \phi(x)}{\partial x} + \frac{1}{2} \text{trace} \left( \Gamma(x) \Gamma(x)' \frac{\partial^2 \phi(x)}{\partial x \partial x'} \right)$$
Eigenfunctions - Eigenvalues

Definition

A Borel function \( \phi \) is an **eigenfunction** of the extended generator \( \mathcal{A} \) with **eigenvalue** \( \rho \) if \( \mathcal{A} \phi = \rho \phi \)

Definition

A **principal eigenfunction** of the extended generator is an eigenfunction that is strictly positive
Proposition

Suppose that $\phi$ is an eigenfunction of the extended generator $\mathbb{A}$ with the associated eigenvalue $\rho$. Then

$$\exp(-\rho t)M_t \phi(X_t)$$

is a local martingale

When $\phi$ is the principal eigenfunction

$$M_t = \exp(\rho t) \hat{M}_t \begin{bmatrix} \phi(X_0) \\ \phi(X_t) \end{bmatrix}$$

where,

$$\hat{M}_t = \exp(-\rho t) M_t \left( \frac{\phi(X_t)}{\phi(X_0)} \right)$$

Further if $\hat{M}_t$ is a martingale,

$$\mathbb{A}_M \phi = \rho \phi \iff \mathbb{M}_t \phi = \exp(\rho t) \phi$$
Discussion: Multiplicative Factorization

\[ M_t = \exp(\rho t) \hat{M}_t \begin{bmatrix} \phi(X_0) \\ \phi(X_t) \end{bmatrix} \]

- Labels:
  - \( \rho \): growth rate
  - \( \hat{M}_t \): martingale component
  - \( \begin{bmatrix} \phi(X_0) \\ \phi(X_t) \end{bmatrix} \): transient or stationary component.

At a fundamental level these methods have allowed us to disentangle the time dependence and state dependence for highly nonlinear functionals.

- Existence
- Uniqueness
- Additive factorization
Proposition

Let $\rho$ and $\phi$ be the principal eigenvalue-eigenfunction pair that stabilizes $X_t$ then,

$$\lim_{t \to \infty} \exp(-\rho t) M_t \psi = \phi \int \frac{\psi}{\phi} d\hat{\zeta}$$

where $\hat{\zeta}$ is the stationary distribution associated with the $\hat{M}$-distorted process $X_t$.

After rescaling the semigroup to eliminate the this growth (or decay), the result says that the limiting state dependence is proportional to the dominant eigenfunction $\phi$ for all alternative functions $\psi$. 
Given a valuation functional $V_t = \langle \beta^\nu, \gamma^\nu \rangle$ and a corresponding $S_t$,

- $\epsilon_\nu(\gamma^\nu)$: Local Risk-Return frontier
- $\rho_\nu(\gamma^\nu)$: Asymptotic Risk-Return frontier
Let $X_t$ be the “Feller-OU” setup and $V_t = \exp(A^v_t)$ and $S_t = \exp(A^s_t)$

$$dX^f_t = \xi_f (\bar{x}_f - X^f_t) dt + \sqrt{X^f_t} \sigma_f dB^f_t$$

$$dX^o_t = \xi_o (\bar{x}_o - X^o_t) dt + \sigma_o dB^o_t$$

$$dA^v_t = \left( \bar{\beta}^v + \beta^v_f X^f_t + \beta^v_o X^o_t \right) dt + \sqrt{X^f_t} \gamma^v_f dB^f_t + \gamma^v_o dB^o_t$$

$$dA^s_t = \left( \bar{\beta}^s + \beta^s_f X^f_t + \beta^s_o X^o_t \right) dt + \sqrt{X^f_t} \gamma^s_f dB^f_t + \gamma^s_o dB^o_t$$
By Ito’s rule,

\[ \frac{dV_t}{V_t} = \left( \beta^\nu(X_t) + \frac{1}{2} \text{trace}[\gamma^\nu(X_t)\gamma^\nu(X_t)'] \right) dt + \gamma(X)dB_t \]

\[ \epsilon^\nu = \beta^\nu + \frac{1}{2} \text{trace}(\gamma^\nu\gamma^{\nu'}) \]

Note the link to the generator,

\[ \epsilon^\nu = A^\nu I \]

where

\[ I(x) = 1 \quad \forall x \]
\[ M = VS = \langle \beta^m, \gamma^m \rangle \]

\[ \beta^m = \beta^v + \beta^s \]
\[ \gamma^m = \gamma^v + \gamma^s \]

\[ M = VS \text{ is a martingale } \Rightarrow \mathbb{A}_M I = 0 \]

\[ \beta^v + \beta^s = -\frac{1}{2} \text{trace} \left[ (\gamma^v + \gamma^s)(\gamma^v + \gamma^s)' \right] \]

\[ \epsilon_v = -\beta^s - \frac{1}{2} \text{trace} (\gamma^s \gamma^{s'}) - \text{trace} (\gamma^s \gamma^{v'}) \]

Risk free rate

Risk compensation
Local Risk Return Trade off

For the parametrization above we have

$$\beta^j(X_t) = \bar{\beta}^j + \beta^j_f X^f_t + \beta^j_o X^o_t \quad j \in \{s, v, m\}$$

$$\gamma^j(X_t) = \begin{bmatrix} \sqrt{X^f_t} \gamma^j_f & 0 \\ 0 & \gamma^j_o \end{bmatrix} \quad j \in \{s, v, m\}$$

$$\epsilon_v(\gamma^v_f, \gamma^v_o) = RF^l + L^l(\gamma^v_f, \gamma^v_o)$$

$$RF^l = -[\bar{\beta}^s + \beta^s_f X^f_t + \beta^s_o X^o_t] - \frac{1}{2}[(X^f_t)(\gamma^s_f)^2 + (\gamma^s_o)^2]$$

$$L^l(\gamma^v_f, \gamma^v_o) = -(X^f_t)(\gamma^s_f)(\gamma^v_f) - (\gamma^s_o)(\gamma^v_o)$$
Asymptotic Risk Return Trade off

Compute \((\phi, \rho)\) for the semigroup generated by \(V_t\)
Guess and Verify Method
\(\phi(x) = \exp(c_f x_f + c_o x_o)\) and solve for

\[ A_v \phi(x) = \rho \phi(x) \]

We get,

\[
c_f(\gamma_f^v) = \frac{(\xi_f - \gamma_f^v \sigma_f) \pm \sqrt{(\xi_f - \gamma_f^v \sigma_f)^2 - \sigma_f^2 (2\beta_f^v + \gamma_f^2)}}{\sigma_f^2}
\]

\[
c_o = \frac{\beta_o}{\xi_o}
\]

\[
\rho = \bar{\beta}^v + \frac{(\gamma_o^v)^2}{2} + c_f \xi_f \bar{x}_f + c_o (\xi_o \bar{x}_o + \gamma_o \sigma_o) + (c_o)^2 \sigma_o^2
\]
Asymptotic Risk Return Trade off

The pricing restriction ($M = VS$ is a martingale) gives us

$$\beta^v(\beta^s, \gamma^s, \gamma^v)$$

$$\rho(\gamma^v_o, \gamma^v_f) = -\beta^s - \left(\frac{\gamma^s_o}{2}\right)^2 - \gamma^s_o \gamma^v_o + c_f \xi_f \bar{x}_f c_o (\xi_o \bar{x}_o + \gamma^v_o \sigma_o) + (c_o)^2 \frac{\sigma^2_o}{2}$$

$$\rho(\gamma^v_f, \gamma^v_f) = RF^{as} + L^{as}(\gamma^v_o) + NL^{as}(\gamma^v_f)$$

where,

$$RF^{as} = -\beta^s - \left(\frac{\gamma^s_o}{2}\right)^2 + c_o \xi_o \bar{x}_o + (c_o)^2 \frac{\sigma^2_o}{2}$$

$$L^{as}(\gamma^v_o) = -\left(\gamma^s_o - \beta^s_o \frac{\sigma_o}{\xi_o}\right) \gamma^v_o$$

$$NL^{as}(\gamma^v_f) = \xi_o \bar{x}_o c_f (\gamma^f_v)$$

Thus we have a very *sharp* characterization of the risk-return frontier,

Local Risk Return Frontier

\[ \epsilon_v(\gamma_f^v, \gamma_o^v) = RF^l + L^l(\gamma_f^v, \gamma_o^v) \]

Asymptotic Risk Return Frontier

\[ \rho(\gamma_f^v, \gamma_f^v) = RF^{as} + L^{as}(\gamma_o^v) + NL^{as}(\gamma_f^v) \]