Long-Term Risk : An Operator Approach

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Motivation

Goals of this paper

1. Introduce operator methods in analysis of dynamic systems
2. Formally describe the mathematical framework to apply operator methods
3. Discuss a few applications in asset pricing
   - Term Structure of Risk
   - Why should we care about it ?

Outline of the structure

Continuous time nonlinear Markov environment

1. Introduce Operator semigroups

$$\mathbb{M}_t\psi(x) = E[M_t\psi(X_t)|X_0 = x]$$

2. Obtain Multiplicative Factorization

$$M_t = \exp(\rho t)\hat{M}_t \frac{\phi(X_0)}{X_t}$$

3. Characterize (ρ, ϕ) as a solution to an eigenvalue - eigenfunction problem

$$\mathbb{M}_t\phi(x) = \exp(\rho t)\phi(x)$$

4. Establish existence and uniqueness of this decomposition

Underlying Markov process

Given a probability space $\{\Omega, \mathcal{F}, P\}$,
Let $\{X_t : t \geq 0\}$ be strong Markov, RCLL with the natural filtration.
Examples
1.D_0 is finite : Finite-State Markov Chain with intensity matrix $\mathbb{U}$
2.D_0 is infinite : Semimartingales with continuous and pure jump components
In particular X_t is an Ito process with following local evolution :

$$dX_t^f = \xi_f(\bar{x}_f - X_t^f)dt + \sqrt{X_t^f}\sigma_f dB_t^f$$

$$dX^o = \xi_o(\bar{x}_o - X_t^o)dt + \sigma_o dB_t^o$$

Functionals

Definition

A **functional** *M is a real-values process* $\{M_t : t \geq 0\}$ *that is adapted and has RCLL paths.*

Definition

A functional M is **multiplicative** *if ,*

1. $M_0 = 1$
2. $M_{t+u} = M_u(\Theta_t)M_t \quad \forall t, u$

Here Θ_τ *is the "shift-operator"*

$$\Theta_\tau(X) = Y \quad Y \equiv \{Y_t = X_{t+\tau} : t \geq 0\}$$

Functionals

Definition

A functional A is **additive** *if ,*

1. $A_0 = 0$
2. $A_{t+u} = A_u(\Theta_t) + A_t \quad \forall t, u$

Definition

A **Valuation functional** $\{V_t : t \geq 0\}$ *is a multiplicative functional such that* $\{V_t S_t : t \geq 0\}$ *is a martingale for some strictly positive multiplicative functional* S_t

Semigroups

Let $\langle L, \|.\| \rangle$ be a Banach space and $\{\mathbb{T}_t : t \geq 0\}$ be a family of operators on L

Definition

A family of linear operators $\{\mathbb{T}_t : t \geq 0\}$ *is a one-parameter* **semigroup** *if ,*

1. $\mathbb{T}_0 = \mathbb{I}$
2. $\mathbb{T}_{t+s} = \mathbb{T}_t \mathbb{T}_s \quad \forall t, s$

Discussion - Functionals and semigroups

- Additive,multiplicative functionals are closed under respective binary operations
- If A is additive, then $M = exp(A)$ is multiplicative.
  Parametrization : Additive functional $A_t = \langle \beta^A, \gamma^A \rangle$

$$A_t = \int_0^t \beta^a(X_u)du + \int_0^t \gamma^a(X_u)dB_u$$

  Further whenever $M = exp(A)$, for simplicity, we would also parameterize M with $\langle \beta^A, \gamma^A \rangle$
- If M is a multiplicative functional such that for all $\psi \in L, \quad E[M_t\psi(X_t)|X_0 = x] \in L$, then $\mathbb{M} \equiv \{\mathbb{M}_t\}$ is a one-parameter semigroup,

$$\mathbb{M}_t = E[M_t\psi(X_t)|X_0 = x]$$

Examples - Functionals and semigroups

1. Stochastic Discount Factors : To the previously described "Feller-OU" setup add the following two things :

- Log consumption : $dc_t = X_t^o dt + \sqrt{X_t^f}\vartheta_f dB_t^f + \vartheta_o dB_t^o$
- Preferences : $E \int_0^\infty \exp(-bt)\frac{C_t^{1-a}-1}{1-a}dt$

The implied stochastic discount factor is $S_t = exp(A_t^s)$ where

$$dA_t^s = (-aX_t^0 - b)dt - a\sqrt{(}X_t^f)\vartheta_f dB_t^f - a\vartheta_o dB_t^o$$

2. Growth Functionals : $D_t = G_t\psi(X_t)D_0$ where G_t is a multiplicative functional
3. Semigroups : Conditional expectation operators

Generators

Definition

A Borel function ψ *belongs to the domain of the* **extended generator** $\mathbb{A}_M$ *of the multiplicative functional* M *if there exists a Borel function* χ *such that* $N_t = M_t\psi(X_t) - \psi(X_0) - \int_0^t M_s\chi(X_s)ds$ *is a local martingale with respect to the filtration* $\{\mathcal{F}_t : t \geq 0\}$
In this case, the extended generator assigns the function ψ *to* χ *and we write* $\mathbb{A}_M\psi = \chi$.

Formalizes the concept of the "expected time derivative".

Generators-Examples

Let $X_t = \langle \xi, \Gamma \rangle$

- $M_t = 1$

$$\mathbb{A}_I \phi(x) = \xi(x) . \frac{\partial \phi(x)}{\partial x} + \frac{1}{2} \text{trace} \left(\Gamma(x) \Gamma(x)' \frac{\partial^2 \phi(x)}{\partial x \partial x'} \right)$$

- For general $M_t = \langle \beta, \gamma \rangle$

$$\begin{aligned} \mathbb{A}_M \phi(x) &= \left(\beta(x) + \frac{|\gamma(x)|^2}{2} \right) \phi(x) \\ &+ \left(\xi(x) + \Gamma(x)\gamma(x) \right) . \frac{\partial \phi(x)}{\partial x} \\ &+ \frac{1}{2} \text{trace} \left(\Gamma(x) \Gamma(x)' \frac{\partial^2 \phi(x)}{\partial x \partial x'} \right) \end{aligned}$$

Eigenfunctions - Eigenvalues

Definition

A Borel function ϕ *is an* **eigenfunction** *of the extended generator* $\mathbb{A}$ *with* **eigenvalue** ρ *if* $\mathbb{A}\phi = \rho\phi$

Definition

A **principal eigenfunction** *of the extended generator is an eigenfunction that is strictly positive*

Multiplicative Factorization

Proposition

Suppose that ϕ is an eigenfunction of the extended generator $\mathbb{A}$ with the associated eigenvalue ρ. Then

$$exp(-\rho t) M_t \phi(X_t)$$

is a local martingale

When ϕ is the principal eigenfunction

$$M_t = exp(\rho t) \hat{M}_t \left[\frac{\phi(X_0)}{\phi(X_t)} \right]$$

where,
$\hat{M}_t = exp(-\rho t) M_t \left(\frac{\phi(X_t)}{\phi(X_0)} \right)$
Further if $\hat{M}_t$ is a martingale,

$$\mathbb{A}_M \phi = \rho \phi \iff \mathbb{M}_t \phi = exp(\rho t) \phi$$

Discussion : Multiplicative Factorization

$$M_t = exp(\rho t)\hat{M}_t \left[\frac{\phi(X_0)}{\phi(X_t)}\right]$$

- Labels :

  ρ - growth rate

  $\hat{M}_t$ - martingale component

  $\left[\frac{\phi(X_0)}{\phi(X_t)}\right]$ - transient or stationary component.

  At a fundamental level these methods have allowed us to disentangle the time dependence and state dependence for highly nonlinear functionals.

- Existence
- Uniqueness
- Additive factorization

Proposition

Let ρ and ϕ be the principal eigenvalue-eigenfunction pair that stabilizes X_t then,

$$\lim_{t\to\infty} \exp(-\rho t)\mathbb{M}_t\psi = \phi \int \frac{\psi}{\phi} d\hat{\varsigma}$$

where $\hat{\varsigma}$ is the stationary distribution associated with the $\hat{M}$-distorted process X_t.

After rescaling the semigroup to eliminate the this growth (or decay), the result says that the limiting state dependence is proportional to the dominant eigenfunction ϕ for all alternative functions ψ

Risk-Return trade off - Local vs Asymptotic

Given a valuation functional $V_t = \langle \beta^v, \gamma^v \rangle$ and a correspondingS_t,

- $\epsilon_v(\gamma^v)$: Local Risk-Return frontier
- $\rho_v(\gamma^v)$: Asymptotic Risk-Return frontier

Underlying Environment

Let X_t be the "Feller-OU" setup and $V_t = \exp(A_t^v)$ and $S_t = \exp(A_t^s)$

$$dX_t^f = \xi_f(\bar{x}_f - X_t^f)dt + \sqrt{X_t^f}\sigma_f dB_t^f$$

$$dX^o = \xi_o(\bar{x}_o - X_t^o)dt + \sigma_o dB_t^o$$

$$dA_t^v = \left(\bar{\beta}^v + \beta_f^v X_t^f + \beta_o^v X_t^o\right) dt + \sqrt{X_t^f}\gamma_f^v dB_t^f + \gamma_o^v dB_t^o$$

$$dA_t^s = \left(\bar{\beta}^s + \beta_f^s X_t^f + \beta_o^s X_t^o\right) dt + \sqrt{X_t^f}\gamma_f^s dB_t^f + \gamma_o^s dB_t^o$$

By Ito's rule,

$$\frac{dV_t}{V_t} = \left(\beta^v(X_t) + \frac{1}{2}\text{trace}[\gamma^v(X_t)\gamma^v(X_t)']\right) dt + \gamma(X) dB_t$$

$$\epsilon_v = \beta^v + \frac{1}{2}\text{trace}(\gamma^v \gamma^{v\prime})$$

Note the link to the generator,

$$\epsilon_v = \mathbb{A}_v I$$

where

$$I(x) = 1 \quad \forall x$$

Local Risk Return Trade off

$M = VS = \langle \beta^m, \gamma^m \rangle$

$$\beta^m = \beta^v + \beta^s$$

$$\gamma^m = \gamma^v + \gamma^s$$

$M = VS$ is a martingale $\Rightarrow \mathbb{A}_M I = 0$

$$\beta^v + \beta^s = -\frac{1}{2}\text{trace}\left[(\gamma^v + \gamma^s)(\gamma^v + \gamma^s)'\right]$$

$$\epsilon_v = \underbrace{-\beta^s - \frac{1}{2}\text{trace}(\gamma^s \gamma^{s\prime})}_{\text{Risk free rate}} - \underbrace{\text{trace}(\gamma^s \gamma^{v\prime})}_{\text{Risk compensation}}$$

Local Risk Return Trade off

For the parametrization above we have

$$\beta^j(X_t) = \bar{\beta}^j + \beta^j_f X^f_t + \beta^j_o X^o_t \quad j \in \{s, v, m\}$$

$$\gamma^j(X_t) = \begin{bmatrix} \sqrt{X^f_t}\gamma^j_f & 0 \\ 0 & \gamma^j_o \end{bmatrix} \quad j \in \{s, v, m\}$$

$$\epsilon_v(\gamma^v_f, \gamma^v_o) = RF^l + L^l(\gamma^v_f, \gamma^v_o)$$

$$RF^l = -[\bar{\beta}^s + \beta^s_f X^f_t + \beta^s_o X^o_t] - \frac{1}{2}[(X^f_t)(\gamma^s_f)^2 + (\gamma^s_o)^2]$$

$$L^l(\gamma^v_f, \gamma^v_o) = -(X^f_t)(\gamma^s_f)(\gamma^v_f) - (\gamma^s_o)(\gamma^v_o)$$

Asymptotic Risk Return Trade off

Compute (ϕ, ρ) for the semigroup generated by V_t
Guess and Verify Method
$\phi(x) = \exp(c_f x_f + c_o x_o)$ and solve for

$$\mathbb{A}_v \phi(x) = \rho \phi(x)$$

We get ,

$$c_f(\gamma_f^v) = \frac{(\xi_f - \gamma_f^v \sigma_f) \pm \sqrt{(\xi_f - \gamma_f^v \sigma_f)^2 - \sigma_f^2 (2\beta_f^v + \gamma_f^2)}}{\sigma_f^2}$$

$$c_o = \frac{\beta_o}{\xi_o}$$

$$\rho = \bar{\beta}^v + \frac{(\gamma_o^v)^2}{2} + c_f \xi_f \bar{x}_f + c_o(\xi_o \bar{x}_o + \gamma_o \sigma_o) + (c_o)^2 \frac{\sigma_o^2}{2}$$

Asymptotic Risk Return Trade off

The pricing restriction ($M = VS$ is a martingale) gives us $\beta^v(\beta^s, \gamma^s, \gamma^v)$

$$\rho(\gamma_o^v, \gamma_f^v) = -\bar{\beta}^s - \frac{(\gamma_o^s)^2}{2} - \gamma_o^s\gamma_o^v + c_f\xi_f\bar{x}_f c_o(\xi_o\bar{x}_o + \gamma_o^v\sigma_o) + (c_o)^2\frac{\sigma_o^2}{2}$$

$$\rho(\gamma_f^v, \gamma_f^v) = RF^{as} + L^{as}(\gamma_o^v) + NL^{as}(\gamma_f^v)$$

where,

$$RF^{as} = -\bar{\beta}^s - \frac{(\gamma_o^s)^2}{2} + c_o\xi_o\bar{x}_o + (c_o)^2\frac{\sigma_o^2}{2}$$

$$L^{as}(\gamma_o^v) = -\left(\gamma_o^s - \beta_o^s\frac{\sigma_o}{\xi_o}\right)\gamma_o^v$$

$$NL^{as}(\gamma_f^v) = \xi_o\bar{x}_o c_f(\gamma_v^f)$$

Risk-Return trade off - Local vs Asymptotic

Thus we have a very *sharp* characterization of the risk-return frontier,

Local Risk Return Frontier

$$\epsilon_v(\gamma_f^v, \gamma_o^v) = RF^l + L^l(\gamma_f^v, \gamma_o^v)$$

Asymptotic Risk Return Frontier

$$\rho(\gamma_f^v, \gamma_f^v) = RF^{as} + L^{as}(\gamma_o^v) + NL^{as}(\gamma_f^v)$$