Predictive Regressions

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Predictive Regressions

$$y_t = \alpha + \beta x_{t-1} + u_t,$$

y_t is a change in an asset's price.
x_{t-1} is a lagged variable related to asset prices.

- $E[u_t | x_{t-1}, x_{t-2}, \ldots] = 0$ and $\hat{\beta}$ (the OLS estimator of β) is consistent.
- However, since $E[\mathbf{u}|\mathbf{x}] \neq 0$, $\hat{\beta}$ is biased.

This paper

1. Finite sampling distribution of the OLS estimator $\hat{\beta}$.
2. Bayesian approach with 'non-informative' priors.
3. Portfolio Problem.

Distributional Assumptions

$$y_t = \alpha + \beta x_{t-1} + u_t,$$

$$x_t = \theta + \rho x_{t-1} + v_t,$$

$(u_t, \nu_t)'$ is *i.i.d.* normal with mean zero and covariance matrix

$$cov\left(\begin{bmatrix} u_t \\ v_t \end{bmatrix}, [u_t \quad v_t]\right) = \Sigma = \begin{bmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{vu} & \sigma_v^2 \end{bmatrix}.$$

It is also assumed that the regressor is stationary $|\rho| < 1$.

Finite-Sample properties of $\hat{\beta}$

$$E(\hat{\beta} - \beta) = \frac{\sigma_{uv}}{\sigma_v^2} E(\hat{\rho} - \rho)$$

The bias in $\hat{\rho}$, to order $1/T$, is $-(1 + 3\rho)/T$. Kendall(1954).

Thus

$$E(\hat{\beta} - \beta) = -\frac{\sigma_{uv}}{\sigma_v^2} \left(\frac{1 + 3\rho}{T} \right) + O(1/T^2).$$

Finite-Sample properties of $\hat{\beta}$

	Sample period			
	1927–1996	1927–1951	1952–1996	1977–1996
A. True properties				
Bias	0.07	0.18	0.18	0.42
Standard deviation	0.16	0.33	0.27	0.45
Skewness	0.71	0.83	0.98	1.29
Kurtosis	3.84	4.14	4.62	5.83
p-value for $\beta = 0$	0.17	0.42	0.15	0.64
B. Properties in the standard regression setting				
Bias	0	0	0	0
Standard deviation	0.14	0.27	0.20	0.30
Skewness	0	0	0	0
Kurtosis	3	3	3	3
p-value for $\beta = 0$	0.06	0.22	0.02	0.26
C. Sample characteristics and parameter values				
$\hat{\beta}$	0.21	0.21	0.44	0.19
T	840	300	540	240
ρ	0.972	0.948	0.980	0.987
$\sigma_u^2 \times 10^4$	30.05	54.46	16.42	17.50
$\sigma_v^2 \times 10^4$	0.108	0.247	0.029	0.033
$\sigma_{uv} \times 10^4$	$-$ 1.621	$-$ 3.360	$-$ 0.651	$-$ 0.715

Bayesian Approach

The posterior for $b = (\alpha\ \beta\ \theta\ \rho)'$ and Σ is given by:

$$p(b, \Sigma | D) \propto L(b, \Sigma; D) p(b, \Sigma),$$

D is the data, which consists of $z = (y'\ x')'$ and the initial observation x_0.

The Bayesian estimator of β is taken to be the mean of the marginal posterior $E(\beta | D)$.

4 different posterior distributions.

- Specification of the likelihood.
- 'non-informative' priors.

Bayesian Approach

The conditional likelihood (assuming x_0 is non-stochastic):

$$\begin{aligned} L_c(b, \Sigma; D) &= p(z|x_o, b, \Sigma) \\ &= (2\pi|\Sigma|)^{-T/2} \exp\{-1/2(z - Zb)'(\Sigma^{-1} \otimes I_T)(z - Zb)\}, \end{aligned}$$

where $Z = I_2 \otimes X$,

The conditional likelihood is maximized at

$$\hat{b} = (\hat{\alpha}\ \hat{\beta}\ \hat{\theta}\ \hat{\rho})' = (Z'Z)^{-1}Z'z$$

Non-informative Jeffrey's priors

$$p(b, \Sigma) = p(b)p(\Sigma) \propto |\Sigma|^{-3/2}$$

The resulting posterior has the property that $E(\beta|D) = \hat{\beta}$, even though, $E(\hat{\beta}) \neq \beta$.

Exact Likelihood and Sharper priors

- Restrict the process x to be stationary $|\rho| < 1$
  Then the joint prior is:

$$p(b, \Sigma) \propto |\Sigma|^{-3/2}, \rho \in (-1, 1).$$

- x_0 is a realization of the same stochastic process generating $x_1, \ldots, x_T$.

$$p(x_0|b, \Sigma) = \left(\frac{1-\rho^2}{2\pi\sigma_v^2}\right)^{1/2} \exp\left\{-\frac{1-\rho^2}{2\sigma_v^2}\left(x_0 - \frac{\theta}{1-\rho}\right)^2\right\}.$$

  The ‘exact’ likelihood function is:

$$L_e(b, \Sigma; D) = p(z, x_0|b, \Sigma) = p(z|x_0, b, \Sigma)p(x_0|b, \Sigma),$$

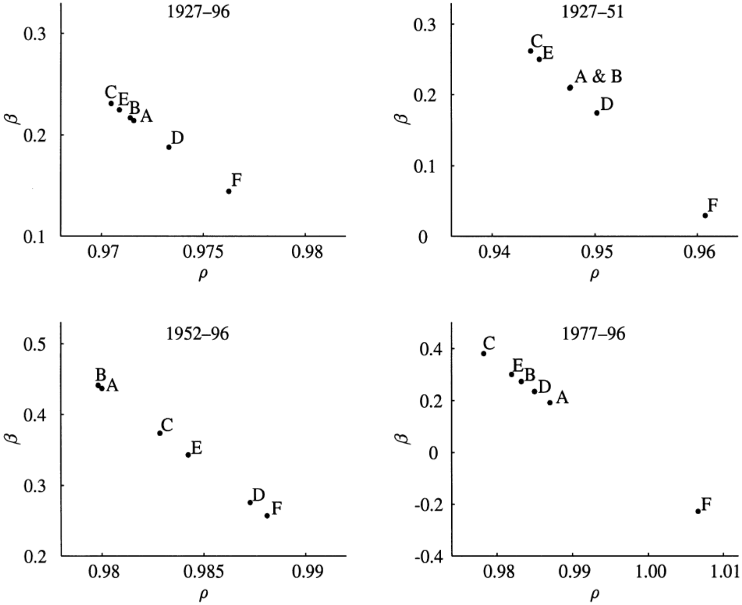
An ’approximate’ Jeffrey’s prior when Z is stochastic is given by:

$$p(b, \Sigma) \propto (1-\rho^2)^{-1}\sigma_v^2|\Sigma|^{-5/2}, \rho \in (-1, 1).$$

Posterior distributions for β

	Sample period			
	1927–1996	1927–1951	1952–1996	1977–1996
A. Conditional likelihood; $p(b, \Sigma) \propto \lvert\Sigma\rvert^{-3/2}, \rho \in (-\infty, \infty)$				
Mean	0.21	0.21	0.44	0.19
Std. Dev.	0.14	0.28	0.20	0.30
Skewness	0	0	0	0
Kurtosis	3.01	3.02	3.01	3.03
Prob($\beta \leqslant 0$)	0.06	0.22	0.02	0.26
B. Conditional likelihood; $p(b, \Sigma) \propto \lvert\Sigma\rvert^{-3/2}, \rho \in (-1, 1)$				
Mean	0.21	0.21	0.44	0.27
Std. Dev.	0.14	0.27	0.20	0.25
Skewness	0.02	0.04	0.12	0.45
Kurtosis	2.98	2.96	2.90	3.04
Prob($\beta \leqslant 0$)	0.06	0.22	0.01	0.13
C. Exact likelihood; $p(b, \Sigma) \propto \lvert\Sigma\rvert^{-3/2}, \rho \in (-1, 1)$				
Mean	0.23	0.26	0.38	0.38
Std. Dev.	0.14	0.26	0.18	0.24
Skewness	0.03	0.08	0.24	0.36
Kurtosis	2.97	2.95	2.93	3.02
Prob($\beta \leqslant 0$)	0.05	0.16	0.01	0.05
D. Exact likelihood; $p(b, \Sigma) \propto (1-\rho^2)^{-1}\sigma_v^2\lvert\Sigma\rvert^{-5/2}, \rho \in (-1, 1)$				
Mean	0.19	0.17	0.28	0.24
Std. Dev.	0.14	0.28	0.18	0.24
Skewness	0.00	0.04	0.37	0.53
Kurtosis	2.95	2.90	2.84	3.18
Prob($\beta \leqslant 0$)	0.10	0.28	0.05	0.16

Estimates of β and ρ



An asset allocation problem

Buy-and-hold investor who allocates wealth between stocks and cash.

$$\max_{\omega} \; E(U(W_{T+K}|D))$$

where

$$U(W_{T+K}) = \frac{1}{\gamma} W_{T+K}^{\gamma}$$

$$W_{T+K} = W_T[\omega \exp\{y_{T+K,(K)} + K i_T\} + (1-\omega)\exp\{K i_T\}].$$

The K-period excess return is given by,

$$Y_{T+K,(K)} = \sum_{k=1}^{K} y_{t+k}.$$

with predictive distribution,

$$p(Y_{T+K,(K)}|D) = \int_{b,\Sigma} p(Y_{T+K,(K)}|b,\Sigma,D)p(b,\Sigma|D)dbd\Sigma$$

Optimal stock allocation

Investment horizon	Current dividend yield (x_T)				
	2%	3%	4%	5%	6%
A. Conditional likelihood; $p(b, \Sigma) \propto \|\Sigma\|^{-3/2}, \rho \in (-\infty, \infty)$					
1 month	0	22	61	97	100
1 yr	0	27	65	100	100
5 yr	11	50	81	86	81
10 yr	37	69	71	63	55
20 yr	63	58	52	44	38
B. Exact likelihood; $p(b, \Sigma) \propto \|\Sigma\|^{-3/2}, \rho \in (-1, 1)$					
1 month	0	15	46	79	100
1 yr	0	18	51	82	100
5 yr	4	37	67	83	85
10 yr	27	57	73	76	71
20 yr	57	65	62	59	54
C. Exact likelihood; $p(b, \Sigma) \propto (1-\rho^2)^{-1}\sigma_v^2\|\Sigma\|^{-5/2}, \rho \in (-1, 1)$					
1 month	0	21	45	68	91
1 yr	1	24	48	70	86
5 yr	13	37	57	68	69
10 yr	29	51	60	60	56
20 yr	50	55	52	47	42
D. Conditional MLEs as true parameters (ignore estimation risk)					
1 month	0	22	60	98	100
1 yr	0	27	68	100	100
5 yr	7	55	100	100	100
10 yr	45	92	100	100	100
20 yr	100	100	100	100	100

Expected excess return

Investment horizon	Current dividend yield (x_T)				
	2%	3%	4%	5%	6%
A. Conditional likelihood; $p(b, \Sigma) \propto \|\Sigma\|^{-3/2}$, $\rho \in (-\infty, \infty)$					
1 month	−3.2	2.0	7.3	12.5	17.7
1 yr	−2.3	2.4	7.0	11.6	16.2
5 yr	0.4	3.2	6.1	9.0	11.9
10 yr	1.9	3.7	5.6	7.4	9.3
20 yr	3.1	4.1	5.2	6.2	7.2
B. Exact likelihood; $p(b, \Sigma) \propto \|\Sigma\|^{-3/2}$, $\rho \in (-1, 1)$					
1 month	−3.5	1.0	5.5	10.0	14.5
1 yr	−2.7	1.4	5.4	9.4	13.5
5 yr	−0.2	2.4	5.1	7.8	10.4
10 yr	1.3	3.1	4.9	6.7	8.5
20 yr	2.7	3.7	4.7	5.8	6.8
C. Exact likelihood; $p(b, \Sigma) \propto (1-\rho^2)^{-1}\sigma_v^2\|\Sigma\|^{-5/2}$, $\rho \in (-1, 1)$					
1 month	−1.4	1.9	5.2	8.5	11.9
1 yr	−0.9	2.1	5.2	8.2	11.2
5 yr	0.7	2.8	5.0	7.1	9.2
10 yr	1.8	3.3	4.8	6.3	7.8
20 yr	2.9	3.8	4.7	5.6	6.5
D. Conditional MLEs as true parameters (ignore estimation risk)					
1 month	−3.2	2.0	7.3	12.5	17.8
1 yr	−2.4	2.3	7.0	11.7	16.4
5 yr	0.0	3.1	6.1	9.2	12.3
10 yr	1.6	3.6	5.6	7.6	9.6
20 yr	2.9	4.0	5.1	6.2	7.3

Standard deviation of the excess return

Investment horizon	Current dividend yield (x_T)				
	2%	3%	4%	5%	6%
A. Conditional likelihood; $p(b, \Sigma) \propto |\Sigma|^{-3/2}, \rho \in (-\infty, \infty)$					
1 month	14.1	14.1	14.1	14.1	14.2
1 yr	13.5	13.1	13.0	13.2	13.7
5 yr	11.4	10.7	10.5	10.7	11.4
10 yr	10.2	9.6	9.3	9.4	9.9
20 yr	10.8	9.7	9.1	9.2	9.9
B. Exact likelihood; $p(b, \Sigma) \propto |\Sigma|^{-3/2}, \rho \in (-1, 1)$					
1 month	14.2	14.2	14.1	14.2	14.2
1 yr	13.6	13.3	13.2	13.4	13.8
5 yr	11.4	10.9	10.8	11.1	11.7
10 yr	9.9	9.6	9.5	9.7	10.2
20 yr	8.9	8.7	8.7	8.8	9.1
C. Exact likelihood; $p(b, \Sigma) \propto (1-\rho^2)^{-1}\sigma_v^2|\Sigma|^{-5/2}, \rho \in (-1, 1)$					
1 month	14.2	14.1	14.1	14.2	14.2
1 yr	13.9	13.5	13.5	13.7	14.1
5 yr	12.5	11.8	11.6	12.0	12.9
10 yr	11.4	10.7	10.6	10.9	11.8
20 yr	10.5	10.1	9.9	10.2	10.9
D. Conditional MLEs as true parameters (ignore estimation risk)					
1 month	14.0	14.0	14.0	14.0	14.0
1 yr	12.8	12.8	12.8	12.8	12.8
5 yr	9.6	9.6	9.6	9.6	9.6
10 yr	7.8	7.8	7.8	7.8	7.8
20 yr	6.5	6.5	6.5	6.5	6.5

Role of Skewness

A third order approximation for expected utility

$$E(U(W_{T+K})) = \frac{W_T^{\gamma}}{\gamma}\exp(\gamma\bar{r})\left(1 + \frac{\gamma}{2}\text{var}(r) + \frac{\gamma^2}{6}E(r-\bar{r})^3 + E(O[(r-\bar{r})^4])\right),$$

where $\bar{r} = E(r)$ (average return on the investor's portfolio).

- The skewness in r is a function of the skewness in $Y_{T+K}, (K)$

Skewness of the excess return

Investment horizon	Current dividend yield (x_T)				
	2%	3%	4%	5%	6%
A. Conditional likelihood; $p(b, \Sigma) \propto |\Sigma|^{-3/2}, \rho \in (-\infty, \infty)$					
1 month	0.0	0.0	0.0	0.0	0.0
1 yr	0.1	0.0	0.0	0.0	− 0.1
5 yr	0.4	0.2	0.0	− 0.2	− 0.4
10 yr	0.8	0.5	0.1	− 0.3	− 0.7
20 yr	6.1	3.1	0.6	− 2.2	− 6.1
B. Exact likelihood; $p(b, \Sigma) \propto |\Sigma|^{-3/2}, \rho \in (-1, 1)$					
1 month	0.0	0.0	0.0	0.0	0.0
1 yr	0.0	0.0	0.0	0.0	− 0.1
5 yr	0.3	0.1	0.0	− 0.2	− 0.3
10 yr	0.3	0.2	0.0	− 0.2	− 0.4
20 yr	0.3	0.2	− 0.1	− 0.2	− 0.4
C. Exact likelihood; $p(b, \Sigma) \propto (1-\rho^2)^{-1}\sigma_v^2|\Sigma|^{-5/2}, \rho \in (-1, 1)$					
1 month	0.0	0.0	0.0	0.0	0.0
1 yr	0.0	0.0	0.0	0.0	− 0.1
5 yr	0.3	0.1	0.0	− 0.2	− 0.4
10 yr	0.5	0.2	− 0.1	− 0.4	− 0.6
20 yr	0.6	0.3	− 0.1	− 0.5	− 0.8
D. Conditional MLEs as true parameters (ignore estimation risk)					
1 month	0.0	0.0	0.0	0.0	0.0
1 yr	0.0	0.0	0.0	0.0	0.0
5 yr	0.0	0.0	0.0	0.0	0.0
10 yr	0.0	0.0	0.0	0.0	0.0
30 yr	0.0	0.0	0.0	0.0	0.0