Asset Pricing with Heterogenous Consumers

Constantinides and Duffie
(JPE 1996)

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Asset Pricing and Euler Equations

The representative agent consumption euler equations from a Lucas (1978) type exchange economy fair poorly in explaining security prices.

- Equity Premium Puzzle, Risk Free Rate Puzzle

Within a representative agent framework there have been many generalizations suggested to help the model perform.

- Nonseperable preferences, transaction costs

Constantinides and Duffie (C&D) work under the combined assumptions of consumer heterogeneity and incomplete consumption insurance.

Previous Literature

C&D are not the first to study the implications of heterogenous agents for asset pricing.

- Previous research suggested the potential enrichment of pricing implications from this approach is small.
- Consumers faced uninsurable income risk, borrowing constraints, transaction costs.
- However, income shocks had very low persistence, so consumers came close to full-insurance behavior through buffer stock saving.
- C&D introduce permanent shocks and show heterogeneity has important pricing implications.

Contribution

- Can the observed joint process of dividends, aggregate income, and prices of securities and bonds be consistent with equillibrium in an economy with incomplete consumption insurance and consumer heterogeneity?
- C&D demonstrate, by construction, the existence of an individual income process, such that the equilibrium security and bond prices match the given security and bond price processes.
- The contribution is theoretical, but provides a set of testable hypotheses.

Asset Markets

- n securities indexed by j, in fixed positive supply, that pay dividend d_{jt} and have ex-dividend price P_{jt}.
  Net aggregate dividend: $D_t = \sum_{j=1}^{n} d_{jt}$.
- Default-free bonds of all maturities less than T, with par value 1 unit of consumption and bond price process: $B_t = (B_{t,t+T}, ..., B_{t,t+1})$.
- At time t, consumer i holds a portfolio of shares of securities: $\theta_t^i = \{\theta_{jt}^i : j = 1, ..., n\}$.
- At time t, consumer i also holds a portfolio of bonds: $b_t^i = \{b_{jt}^i : j = n+1, ..., n+T\}$.
- $\theta_{-1}^i = \theta_{-1}^k \ \forall (i,k)$ and $b_{-1}^i = 0 \ \forall i$.

Consumer Problem

Controls:

$$(\theta^i, b^i, C^i) \tag{1}$$

Preferences:

$$E\left[\sum_{t=0}^{\infty} e^{-\rho t}\frac{(C_t^i)^{1-\alpha}}{1-\alpha}\right] \tag{2}$$

Constraints:

$$C_t^i + \theta_t^i P_t + b_t^i B_t = I_t^i + \theta_{t-1}^i (P_t + d_t) + b_{t-1}^i \widehat{B}_t \tag{3}$$

Equilibrium

Given: dividend process d, aggregate income process I, α, ρ.

An Equilibrium is:

- Security and Bond price processes (P, B)
- Given (P, B), optimal consumer policies (θ^i, b^i, C^i)
- $\sum_{i \in A} \theta^i_{jt} = 1$ and $\sum_{i \in A} b^i_{jt} = 0 \ \forall j, t$
- $\sum_{i \in A} C^i_t = C_t := I_t + D_t$

C&D will try to find a (θ^i, b^i, C^i) and I^i to support any given $(P, B, d, I, \alpha, \rho)$.

Individual Labor Income

$$I_t^i = \delta_t^i C_t - D_t$$

$$\delta_t^i = \exp\left[\sum_{s=1}^{t}\left(\eta_s^i y_s - \frac{y_s^2}{2}\right)\right]$$

$$y_t = \left(\frac{2}{\alpha^2+\alpha}\right)^{1/2}\left[\ln\left(\frac{M_t}{M_{t-1}}\right) + \rho + \alpha \ln\left(\frac{C_t}{C_{t-1}}\right)\right]^{1/2}$$

$$\eta_s^i \sim N(0,1)\ , i.i.d.$$

Note: $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \delta_t^n = 1$

Pricing Kernel

Given no-arbitrage condition, a pricing kernel M exists such that $\forall t \; M_t$ satisfies:

$$P_{jt} = \frac{1}{M_t} E_t \left[\sum_{s=t+1}^{\infty} d_{jt} M_s \right] \; , j = 1, \ldots, n$$

$$B_{t,t+s} = \frac{1}{M_t} E_t \left[M_{t+s} \right] \; , s = 1, \ldots, T$$

Pricing Kernel Conditions:

$$\lim_{t \to \infty} E\left[M_t\right] = 0$$

$$\frac{M_{t+1}}{M_t} \geq e^{-\rho} \left(\frac{C_{t+1}}{C_t} \right)^{-\alpha} \; , t \geq 0$$

No Trade Equilibrium

There exists an equilibrium with no trade that supports the given price processes of the securities and bonds.

- For consumer i, calculate MRS with no trade.
- For consumer i, calculate private valuation of security j, $\widehat{P}_{jt}(i)$, under these MRS.
- Show $\widehat{P}_{jt}(i) = P_{jt}$,the given price process.
- Then no trade is an equilibrium and this equilibrium supports the given price processes.

Informal Proof: No Trade 1

$$
\begin{aligned}
MRS^i &= e^{-\rho}\left(\frac{C^i_{t+1}}{C^i_t}\right)^{-\alpha} \\
&= e^{-\rho}\left(\frac{I^i_{t+1}+D_{t+1}}{I^i_t+D_t}\right)^{-\alpha} \\
&= e^{-\rho}\left(\frac{\delta^i_{t+1}C_{t+1}}{\delta^i_t C_t}\right)^{-\alpha} \\
&= e^{-\rho}\left(\frac{C_{t+1}}{C_t}\right)^{-\alpha}\exp\left[-\alpha\left(\eta^i_{t+1}y_{t+1}-\frac{y^2_{t+1}}{2}\right)\right]
\end{aligned}
$$

Informal Proof: No Trade 2

$$\widehat{P}_{jt}(i) := E_t\left[(P_{j,t+1} + d_{j,t+1})MRS^i\right]$$
$$= E_t\left[(P_{j,t+1} + d_{j,t+1})e^{-\rho}\left(\frac{C_{t+1}}{C_t}\right)^{-\alpha} Z_t^i\right]$$

where

$$Z_t^i := E_t\left[\exp\left[-\alpha\left(\eta_{t+1}^i y_{t+1} - \frac{y_{t+1}^2}{2}\right)\right]|y_{t+1}\right]$$
$$= \exp\left[\frac{\alpha(\alpha+1)}{2}y_{t+1}^2\right]$$
$$= e^{-\rho}\left(\frac{C_{t+1}}{C_t}\right)^{-\alpha}\left(\frac{M_{t+1}}{M_t}\right)$$

Informal Proof: No Trade 3

Plug Z_t^i back into equation for $\widehat{P}_{jt}(i)$:

$$\widehat{P}_{jt}(i) = E\left[(P_{j,t+1} + d_{j,t+1})\frac{M_{t+1}}{M_t}\right]$$

Finally, by defintion of M_t,

$$P_{jt} = E\left[(P_{j,t+1} + d_{j,t+1})\frac{M_{t+1}}{M_t}\right]$$

Thus, $\widehat{P}_{jt}(i) = P_{jt}$.

Conclusion

- There exists a no trade equilibrium: $\theta^i = \theta^i_{-1}$, $b^i = 0$, $C^i = \delta^i C$, with I_i defined above.
- Any given (P, B) is supported by this equilibrium.

Interpretation

- No-trade implication is counter-factual.
- C&D suggest we think of C_t^i as post-trade allocations.
- I_t^i constructed just to get closed form solution $C_t^i = \delta_t^i C_t$
- However, C&D have highlighted the rich implications a heterogenous agent model can have on asset pricing.

Main Implication

Consumption Euler Equation of consumer *i* for security *j* contains cross-sectional component:

$$E\left[R_{j,t+1}e^{-\rho}\left(\frac{C_{t+1}}{C_t}\right)^{-\alpha}\exp\left(\frac{\alpha(\alpha+1)}{2}y_{t+1}^2\right)\right]=1$$

Special Cases

- No heterogeneity, $y^2_{t+1} = 0$, yields standard representative consumer Euler Equation.
- If $y^2_{t+1} = a + b \ln\left(\frac{C_{t+1}}{C_t}\right)$ then,

$$E_t\left[R_{j,t+1} e^{-\widetilde{\rho}} \left(\frac{C_{t+1}}{C_t}\right)^{-\widetilde{\alpha}}\right] = 1$$

$$\widetilde{\rho} = \rho - \frac{\alpha(\alpha+1)}{2} a$$

$$\widetilde{\alpha} = \alpha - \frac{\alpha(\alpha+1)}{2} b$$

General Case

Let $R_{F,t+1}$ be the return on a one period risk free bond. Then excess return of security j is:

$$E_t\left[R_{j,t+1}\right] - R_{F,t+1} = -\frac{cov_t(R_{j,t+1}, H_{t+1})}{E_t\left[H_{t+1}\right]}$$

where

$$H_{t+1} = \left(\frac{C_{t+1}}{C_t}\right)^{-\alpha} \exp\left[\frac{\alpha(\alpha+1)}{2} y_{t+1}^2\right]$$

Cross Sectional Standard Deviation

y_t is cross sectional standard deviation of consumption growth.

$$\frac{C_t^i}{C_{t-1}^i} = \frac{\delta_t^i}{\delta_{t-1}^i}\frac{C_t}{C_{t-1}} = \exp\{\eta_t^i y_t - .5y_t^2\}\frac{C_t}{C_{t-1}}$$

Conditioning on C_t

$$\ln\left(\frac{C_t^i/C_t}{C_{t-1}^i/C_{t-1}}\right) = \eta_t^i y_t - .5y_t^2 \sim N(-.5y_t^2, y_t^2)$$