A Theory of Financial Constraints and Firm Dynamics
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Model

- Discrete time, infinite horizon. \( t = 0, 1, \ldots \)
- 2 classes of risk neutral individuals with discount factor \( \delta \)
  - Entrepreneurs/Borrowers: endowment \( M > 0 \), access to a risky project at \( t = 0 \).
  - Banks/Lenders: unlimitted endowment
Risky Project

- Initial investment at $t = 0$, $I_0 > M$
- Per period investment of working capital $k_t$
- Per period return depends on a random variable $\theta \in \Theta = \{L, H\}$ where $\Pr(\theta = H) = p$.

$$\text{Return} = \begin{cases} 
0 & \text{if } \theta = L \\
R(k) > 0 & \text{if } \theta = H
\end{cases}$$

- Revenue outcome is private information for the entrepreneur
- Liquidation generates a scrap value $S > 0$
Contract

- Perfect enforcement environment.
- Take-it-or leave-it offer from the bank to the entrepreneur.
- Let $h^t = \left\{ \hat{\theta}_{\tau} \right\}_{\tau=0}^t$ be the history of reports $\hat{\theta}_t$ up to period $t$.

**Definition**

A contract $\sigma = \left\{ \alpha_t (h^{t-1}), Q_t (h^{t-1}), k_t (h^{t-1}), \tau_t (h^t) \right\}_{t=0}^\infty$ specifies:
- contingent liquidation probabilities $\alpha_t (h^{t-1})$
- transfers $Q_t$ from the lender to the borrower in case of liquidation
- capital advancements $k_t$
- payments $\tau_t$ from the borrower to the lender in case of no liquidation.
Feasible contract

Definition
A contract $\sigma$ is feasible if $\forall t \geq 1$ and $\forall h^{t-1} \in \Theta^{t-1}$ we have

- $\alpha_t (h^{t-1}) \in [0, 1]$
- $Q_t \geq 0$
- $\tau_t (h^{t-1}, H) \leq R (k_t (h^{t-1}))$
- $\tau_t (h^{t-1}, L) \leq 0$
Timing

Lender advances $k_t(h^{t-1})$

Entrepreneur pays $\tau_t(h^{t-1}, \hat{\theta}_t)$

Continue

$1 - \alpha_t(h^{t-1})$

Nature draws $\theta_t$

Entrepreneur reports $\hat{\theta}_t$

Liquidate

$\alpha_t(h^{t-1})$

Entrepreneur receives $Q_t(h^{t-1})$

Lender receives $S - Q_t(h^{t-1})$

Figure 1: The Timing.
Value Functions

- Expected discounted cash flows from contract $\sigma$ and reporting strategy $\hat{\theta} = \left\{ \hat{\theta}_t \left( \theta^t \right) \right\}_{t=1}^{\infty}$
  - for the entrepreneur: $V_t \left( \hat{\theta}, \sigma, h^{t-1} \right) \implies \text{Equity}$
  - for the bank: $B_t \left( \hat{\theta}, \sigma, h^{t-1} \right) \implies \text{Debt}$
Definition
A contract $\sigma$ is **incentive compatible** (IC) if $\forall \hat{\theta}$

$$V_1(\theta, \sigma, h^0) \geq V_1(\hat{\theta}, \sigma, h^0)$$
Optimal contract

\[ \mathcal{V} = \{ V | \exists \sigma \text{ feasible and IC such that } V_1 (\theta, \sigma, h^{t-1}) = V \} \]

**Definition**

For any \( V \in \mathcal{V} \), an optimal contract is the feasible and incentive compatible contract that maximizes the value obtained by the lender while delivering an initial value \( V \) to the entrepreneur.

\[ B (V) = \sup \{ B | \exists \sigma \text{ s.t. } V_1 (\theta, \sigma, h^{t-1}) = V \text{ and } B_1 (\theta, \sigma, h^{t-1}) = B \} \]

Capital structure \((V, B(V))\) implies a value of the firm of \( W (V) = B (V) + V \)
Benchmark: Symmetric information

- Maximize total surplus from the match
  \[ \implies \text{Each period the bank lends the entrepreneur the uncostrained optimal level of working capital } k^* \]
- Any division of surplus is feasible.
- Value of the firm is \( \tilde{W} \)
Private information: Recursive Formulation

- The value of equity summarizes all information provided by the history itself
  \[ V = p \left( R(k) - \tau \right) + \delta \left[ pV^H + (1 - p) V^L \right] \]

- IC becomes
  \[ R(k) - \tau + \delta V^H \geq R(k) + \delta V^L \]
  \[ \tau \leq \delta \left( V^H - V^L \right) \]
Firm Value

\( \hat{W} \) : value of the firm contingent on no liquidation
\( W \) : value of the firm prior to the liquidation decision.

\[
\hat{W}(V) = \max_{k, \tau, V^H, V^L} \left[ p R(k) - k \right] + \delta \left[ p W(V^H) + (1 - p) W(V^L) \right]
\]

subject to

\[
\begin{align*}
V &= p(R(k) - \tau) + \delta \left[ p V^H + (1 - p) V^L \right] \\
\tau &\leq \delta (V^H - V^L) \\
\tau &\leq R(k) \\
V^H, V^L &\geq 0
\end{align*}
\]
Firm value prior to liquidation decision

\[ W(V) = \max_{\alpha \in [0,1], Q, V_c} \alpha S + (1 - \alpha) \hat{W}(V_c) \]

subject to

\[ \alpha Q + (1 - \alpha) V_c = V \]
\[ Q, V_c \geq 0 \]
Firm value

Figure 2: The Value Function.
If $V \in [V_r, \tilde{V}]$, \( \tau = R(k) \) and

\[
W(V) = \max_{k, \tau, V^H, V^L} [pR(k) - k] + \delta \left[ pW(V^H) + (1 - p) W(V^L) \right]
\]

subject to

\[
V = \delta \left[ pV^H + (1 - p) V^L \right]
\]

\[
R(k) \leq \delta \left( V^H - V^L \right)
\]

\[
V^H, V^L \geq 0
\]

Trade off between increasing $k$ and a mean preserving spread of expected value. (since $W$ is concave)
Evolution of Equity

If $V_r < V < \tilde{V}$, $V_t$ is a submartingale with 2 absorbing sets, $V_t = 0$ and $V_t \geq \tilde{V}$

Figure 3: The Dynamics of Equity.
Firm dynamics

Figure 5: Dynamics of Growth and Survival.
Firm value prior to liquidation decision

- 3 regions
  - $V < V_r$ liquidation is possible. Lottery $Q = 0$ with prob $\alpha(V)$ and $V_r$ otherwise
  - $V \in \left[ V_r, \tilde{V} \right)$, no liquidation, $k < k^*$. The entrepreneur is borrowing constraint. $\tau = R(k) = \delta \left( V^H - V^L \right)$ (no dividends paid + Binding IC)
  - $V \geq \tilde{V}$, no liquidation, $k = k^*$. Transfers are not determined.