On Efficient Distribution with Private Information
by Atkeson and Lucas - RES 1992

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The model

- Discrete time, infinite horizon $t = 0, 1, 2, \ldots$
- Single, non storable good
- Continuum of consumers (measure 1):
  - Preferences
    \[
    E \left( \sum_{t=0}^{\infty} (1 - \beta) \beta^t V(c_t) \theta_t \right)
    \]
    where $V' > 0$, $V'' < 0$
  - Idiosyncratic, serially independent, iid taste shock: $
  \theta_t \in \Theta = \{\theta_1, \ldots, \theta_n\}$, and
  \[
  \Pr(\theta_t = \theta_i) = \mu(\theta_i) \text{ for all } i \text{ and all } t
  \]
  - Idiosyncratic taste shocks are private information of the consumers.
Endowments

- Endowment economy: Fixed aggregate endowment $y$ in every period.
- Each consumer enters the economy with a number $w \in D \subseteq \mathbb{R}$ which can be thought of as his initial entitlement to expected, discounted utility.
- Distribution of utilities:

$$\Pr (w \in A) = \psi (A) \text{ for all } A \subseteq D$$

Fraction of people in the economy with $w \in A$. $\psi \in M$, where $M$ is the set of all probability measures on $(D, \mathcal{D})$ where $\mathcal{D}$ are the Borel subsets of $D$. 


Planner’s Objective

Use the minimum amount of resources (as constant, perpetual, endowment flow) to attain a certain distribution $\psi$ given the information available.
A **reporting strategy** is a sequence $z = \{ z_t (\theta^t) \}^\infty_{t=0}$, $z_t : \Theta^{t+1} \to \Theta$

A **truthful** reporting strategy is a reporting strategy $z^* = \{ z^*_t (\theta^t) \}^\infty_{t=0}$ such that $z^*_t (\theta^t) = \theta_t$ for all $t$ and all $\theta^t \in \Theta^{t+1}$

- Reporting history: $z^t = \{ z_0 (\theta_0) , \ldots , z_t (\theta_t) \}$
Plan

Let, \( u_t(w, z^t) = V(c_t(w, z^t)) \), \( u_t: D \times \Theta^{t+1} \rightarrow \mathbb{R} \)

A plan is a sequence \( u = \{u_t(w, z^t)\}_{t=0}^{\infty} \) such that

\[
\lim_{t \to \infty} \beta^t \sum_{s=0}^{\infty} \beta^s u_{t+s}(w, z^{t+s}) \theta_{t+s} = 0
\]

\( \Longrightarrow \) Discounted expected lifetime utility of plan \( u \) and reporting strategy \( z \)

\[
U(w, u, z) \equiv E \left( \sum_{t=0}^{\infty} (1 - \beta) \beta^t u(w, z^t) \theta_t \right)
\]
A plan \( u \) is an **allocation** if it satisfies

- **Truth Telling:** given \( u \) truth telling is optimal for the agents

\[
U(w, u, z^*) \geq U(w, u, z) \quad \text{for all } z \in Z \text{ and all } w \in D
\]

- **Promise keeping:** \( u \) delivers the expected discounted utility to each agent \( w \)

\[
U(w, u, z^*) = w \quad \text{for all } w
\]
Efficiency

An allocation is **efficient** if it attains a distribution $\psi$ at the minimum cost (as constant, perpetual, endowment flow) of attaining $\psi$. 
Planner’s Problem

Objective: Minimize constant stream of endowment to attain a welfare distribution $\psi$.

$$\min_u y$$

such that

- $u$ is an allocation
- $u$ attains $\psi$ with resources $y$, i.e.,

$$\int_{D \times \Theta} C(u_t(w, \theta^t)) \, d\mu(\theta_t) \, d\psi(w) \leq y \text{ for all } t$$

where $C(u_t(w, z^t)) \equiv c_t(w, z^t)$ (inverse function of $V$)
Allocation rule

Let \( f_t (w_t(w_0, z^{t-1}), z_t) = u_t (w_0, z^t), g_t (w_t, z_t) = w_{t+1} \)

A sequence \( \sigma = \{ f_t, g_t \}_{t=0}^{\infty} \) is an allocation rule if it satisfies

(i) Promise keeping: for all \( w_t \) and all \( t \)

\[
\int_{\Theta} [(1 - \beta) f_t (w_t, \theta_t) \theta_t + \beta g_t (w_t, \theta_t)] d\mu = w_t
\]

(ii) Truth telling: for all \( z_t \in \Theta \), all \( w_t \in D \) and all \( t \)

\[
[(1 - \beta) f_t (w_t, \theta_t) \theta_t + \beta g_t (w_t, \theta_t)] d\mu \geq [(1 - \beta) f_t (w_t, z_t) \theta_t + \beta g_t (w_t, z_t)] d\mu
\]

(iii) Boundedness: \( \lim_{t \to \infty} \beta^t g_t (w_t(w_0, z^{*t-1}), z^*_t) = 0 \)
**Planner’s Problem**

\[
\min_{\sigma} y \\
\text{subject to:}
\]

- \(\sigma\) is an allocation rule
- The allocations rule \(\sigma\) attains \(\psi\) with resources \(y\)

\[
\int_{D \times \Theta} C(f_t(w_t, \theta)) \, d\mu(\theta) \, d\psi_t(w) \leq y \text{ for all } t
\]

where for all \(t\)

\[
\psi_{t+1}(w) = S_g(\psi_t)(w) \equiv \int_{D \times \Theta} I \{ w_t : g_t(w_t, \theta) = w \} \, d\mu \, d\psi_t
\]
Equivalence

**Lemma 3.1** Let \( \psi \in M \) and suppose the allocation \( u \) attains \( \psi \) with resources \( y \). Then there is an allocation rule \( \sigma \) that attains \( \psi \) with resources \( y \).

**Lemma 3.2** Let \( \psi \in M \). Suppose the allocation rule \( \sigma \) attains \( \psi \) with resources \( y \) and that \( u \) is the utility plan generated by \( \sigma \). Then \( u \) is an allocation, and \( u \) attains \( \psi \) with resources \( y \).

\[ f_t (w_t(w_0, z^{t-1}), z_t) = u_t (w_0, z^t), g_t (w_t, z_t) = w_{t+1} \]
Planner’s Problem

\[ \min_{\sigma} y \]

subject to:

- \( \sigma \) is an allocation rule
- The allocations rule \( \sigma \) attains \( \psi \) with resources \( y \)

\[
\max_t \left[ \int_{D \times \Theta} C(f_t(w_t, \theta)) \, d\mu(\theta) \, d\psi_t(w) \right] \leq y
\]

where for all \( t \)

\[
\psi_{t+1}(w) = S_g(\psi_t)(w)
\]
Let $\varphi^* (\psi)$ be the minimum cost of attaining a distribution $\psi$.

\[
\varphi (\psi) = \inf_{f,g \in B} \max \left\{ \int_{D \times \Theta} C (f (w, \theta)) \, d\mu (\theta) \, d\psi (w), \varphi^* (S_g \psi) \right\}
\]

where $B$ is the set of all functions $f, g$ such that they satisfy promise keeping and truth telling.
Problem T

Let $X$ be the set of all functions $\varphi : M \rightarrow \mathbb{R}_+ \cup \{+\infty\}$. Define the operator $T : X \rightarrow X$ as

$$(T \varphi)(\psi) = \inf_{f,g \in B} \max \left\{ \int_{D \times \Theta} C(f(w,\theta)) \ d\mu(\theta) \ d\psi(w), \varphi(S_g \psi) \right\}$$

**Lemma 4.1** $\varphi^*$ is a fixed point of $T$
Lemma 4.2 Suppose there are functions $\varphi_a$, $\varphi_c$ and $\varphi$ such that for all $\psi \in M$

- $\varphi_c < \varphi^* < \varphi_a$
- $\lim_{n \to \infty} T^n \varphi_a = \lim_{n \to \infty} T^n \varphi_c = \varphi$

Then, $\varphi = \varphi^*$.

Candidates:

- Autarky: $\varphi_a (\psi) = \int_D C(w) \, d\psi$
- Complete insurance: $\varphi_c (\psi) = \int_{D \times \Theta} C(u_c(w, \theta)) \, d\psi \, d\varphi$

where $u_c(w, \theta)$ is the period utility an agent $w$ with shock $\theta$ would receive if the shock was observable.
Example: log utility

- $V(x) = \log(x)$, $C(u) = \exp(u)$
- $D = \mathbb{R}$
- Bounding functions
  - $\varphi_a(\psi) = \int_D \exp(w) d\psi$
  - $\varphi_c(\psi) = \alpha \int_D \exp(w) d\psi$ where $\alpha = \exp\{-E[\theta \log \theta]\}$

Result: $f, g$ linear in $w$ and independent of $\psi$

- $f(w, \theta) = r(\theta; \alpha) + w$
- $g(w, \theta) = h(\theta; \alpha) + w$

$\implies$ Can choose $r$ and $h$ instead of $f$ and $g$
Dynamics (log utility)

- Evolution of individual promised values:

\[ w_t (w_0, \theta^{t-1}) = w_0 + \sum_{i=0}^{t-1} h(\theta_i; \alpha) \]

- Evolution of period utilities

\[ u_t (w_0, \theta^t) = w_0 + \sum_{i=0}^{t-1} h(\theta_i; \alpha) + r(\theta_t) \]

\[ \implies \text{Cross sectional variance} \]

\[ Var [w_t (w_0, \theta^{t-1})] = Var (w_0) + tVar [h(\theta; \alpha)] \]

\[ Var [u_t (w_0, \theta^t)] = Var (w_0) + tVar [h(\theta; \alpha)] + Var [r(\theta)] \]

The degree of inequality grows without bound when resources are efficiently allocated.