On Efficient Distribution with Private Information

by Atkeson and Lucas - RES 1992

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The model

- Discrete time, infinite horizon $t = 0, 1, 2, ...$
- Single, non storable good
- Continuum of consumers (measure 1):
  - Preferences
$$E\left(\sum_{t=0}^{\infty}(1-\beta)\beta^{t}V(c_t)\theta_t\right)$$
    where $V' > 0$, $V'' < 0$
  - Idiosyncratic, serially independent, iid taste shock: $\theta_t \in \Theta = \{\theta_1, ..., \theta_n\}$, and
$$\Pr(\theta_t = \theta_i) = \mu(\theta_i) \text{ for all } i \text{ and all } t$$
- Idiosyncratic taste shocks are private information of the consumers.

Endowments

- Endowment economy: Fixed aggregate endowment y in every period.
- Each consumer enters the economy with a number $w \in D \subset \mathbb{R}$ which can be thought of as his initial entitlement to expected, discounted utility.
- Distribution of utilities:

$$\Pr(w \in A) = \psi(A) \text{ for all } A \subseteq D$$

Fraction of people in the economy with $w \in A$. $\psi \in M$, where M is the set of all probability measures on $(D, \mathbf{D})$ where $\mathbf{D}$ are the Borel subsets of D.

Planner's Objective

Use the minimum amount of resources (as constant, perpetual, endowment flow) to attain a certain distribution ψ given the information available.

Reporting Strategies

A **reporting strategy** is a sequence $z = \left\{ z_t \left(\theta^t \right) \right\}_{t=0}^{\infty}$, $z_t : \Theta^{t+1} \longrightarrow \Theta$

A **truthful** reporting strategy is a reporting strategy $z^* = \left\{ z_t^* \left(\theta^t \right) \right\}_{t=0}^{\infty}$ such that $z_t^* \left(\theta^t \right) = \theta_t$ for all t and all $\theta^t \in \Theta^{t+1}$

- Reporting history: $z^t = \left\{ z_0 \left(\theta_0 \right), ..., z_t \left(\theta_t \right) \right\}$

Plan

Let, $u_t\left(w, z^t\right) = V\left(c_t\left(w, z^t\right)\right)$, $u_t : D \times \Theta^{t+1} \to \mathbb{R}$

A **plan** is a sequence $u = \left\{u_t\left(w, z^t\right)\right\}_{t=0}^{\infty}$ such that

$$\lim_{t\to\infty} \beta^t \sum_{s=0}^{\infty} \beta^s u_{t+s}\left(w, z^{t+s}\right)\theta_{t+s} = 0$$

$\Longrightarrow$ Discounted expected lifetime utility of plan u and reporting strategy z

$$U\left(w, u, z\right) \equiv E\left(\sum_{t=0}^{\infty}\left(1-\beta\right)\beta^t u\left(w, z^t\right)\theta_t\right)$$

Allocation

A plan u is an **allocation** if it satisfies

- Truth Telling: given u truth telling is optimal for the agents

$$U(w, u, z^*) \geq U(w, u, z) \quad \text{for all } z \in Z \text{ and all } w \in D$$

- Promise keeping: u delivers the expected discounted utility to each agent w

$$U(w, u, z^*) = w \text{ for all } w$$

Efficiency

An allocation is **efficient** if it attains a distribution ψ at the minimum cost (as constant, perpetual, endowment flow) of attaining ψ.

Planner's Problem

Objective: Minimize constant stream of endowment to attain a welfare distribution ψ.

$$\min_u y$$

such that

- u is an allocation
- u attains ψ with resources y, i.e.,

$$\int_{D\times\Theta} C\left(u_t\left(w,\theta^t\right)\right) d\mu\left(\theta_t\right) d\psi\left(w\right) \leq y \quad \text{for all } t$$

where $C\left(u_t\left(w,z^t\right)\right) \equiv c_t\left(w,z^t\right)$ (inverse function of V)

Allocation rule

Let $f_t\left(w_t(w_0, z^{t-1}), z_t\right) = u_t\left(w_0, z^t\right)$, $g_t\left(w_t, z_t\right) = w_{t+1}$

A sequence $\sigma = \{f_t, g_t\}_{t=0}^{\infty}$ is an **allocation rule** if it satisfies
(i) Promise keeping: for all w_t and all t

$$\int_{\Theta} \left[(1-\beta) f_t\left(w_t, \theta_t\right) \theta_t + \beta g_t\left(w_t, \theta_t\right)\right] d\mu = w_t$$

(ii) Truth telling: for all $z_t \in \Theta$, all $w_t \in D$ and all t

$$\left[(1-\beta) f_t\left(w_t, \theta_t\right) \theta_t + \beta g_t\left(w_t, \theta_t\right)\right] d\mu \geq \left[(1-\beta) f_t\left(w_t, z_t\right) \theta_t + \beta g_t\left(w_t, z_t\right)\right] d\mu$$

(iii) Boundedness: $\lim_{t\to\infty} \beta^t g_t\left(w_t(w_0, z^{*t-1}), z_t^*\right) = 0$

Planner's Problem

$$\min_{\sigma} y$$

subject to:

- σ is an allocation rule
- The allocations rule σ attains ψ with resources y

$$\int_{D\times\Theta} C\left(f_t\left(w_t,\theta\right)\right) d\mu\left(\theta\right) d\psi_t\left(w\right) \leq y \text{ for all } t$$

where for all t

$$\psi_{t+1}\left(w\right) = S_g\left(\psi_t\right)\left(w\right) \equiv \int_{D\times\Theta} I\left\{w_t : g_t\left(w_t,\theta\right) = w\right\} d\mu d\psi_t$$

Equivalence

Lemma 3.1 Let $\psi \in M$ and suppose the allocation u attains ψ with resources y. Then there is an allocation rule σ that attains ψ with resources y.

Lemma 3.2 Let $\psi \in M$. Suppose the allocation rule σ attains ψ with resources y and that u is the utility plan generated by σ. Then u is an allocation, and u attains ψ with resources y.

$\Longrightarrow$ We can work with allocation rules σ and then find the corresponding allocation u.

$$f_t\left(w_t(w_0, z^{t-1}), z_t\right) = u_t\left(w_0, z^t\right), g_t\left(w_t, z_t\right) = w_{t+1}$$

Planner's Problem

$$\min_{\sigma} y$$

subject to:

- σ is an allocation rule
- The allocations rule σ attains ψ with resources y

$$\max_{t} \left[\int_{D \times \Theta} C \left(f_t \left(w_t, \theta \right) \right) d\mu \left(\theta \right) d\psi_t \left(w \right) \right] \leq y$$

where for all t

$$\psi_{t+1} \left(w \right) = S_g \left(\psi_t \right) \left(w \right)$$

Bellman Equation

Let $\varphi^*(\psi)$ be the minimum cost of attaining a distribution ψ.

$$\varphi(\psi) = \inf_{f,g \in B} \max\left\{ \int_{D \times \Theta} C\left(f\left(w, \theta\right)\right) d\mu\left(\theta\right) d\psi\left(w\right), \varphi^*\left(S_g \psi\right) \right\}$$

where B is the set of all functions f, g such that they satisfy promise keeping and truth telling.

Problem T

Let X be the set of all functions $\varphi : M \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$. Define the operator $T : X \longrightarrow X$ as

$$(T\varphi)(\psi) = \inf_{f,g \in B} \max \left\{ \int_{D \times \Theta} C(f(w,\theta))\, d\mu(\theta)\, d\psi(w), \varphi(S_g \psi) \right\}$$

Lemma 4.1 φ^* is a fixed point of T

Problem T

Lemma 4.2 Suppose there are functions φ_a, φ_c and φ such that for all $\psi \in M$

- $\varphi_c < \varphi^* < \varphi_a$
- $\lim_{n\to\infty} T^n \varphi_a = \lim_{n\to\infty} T^n \varphi_c = \varphi$

Then, $\varphi = \varphi^*$.

Candidates:

- Autarky: $\varphi_a(\psi) = \int_D C(w)\, d\psi$
- Complete insurance: $\varphi_c(\psi) = \int_{D\times\Theta} C(u_c(w,\theta))\, d\psi d\varphi$, where $u_c(w,\theta)$ is the period utility an agent w with shock θ would receive if the shock was observable.

Example: log utility

- $V(x) = \log(x)$, $C(u) = \exp(u)$
- $D = \mathbb{R}$
- Bounding functions
  - $\varphi_a(\psi) = \int_D \exp(w)\, d\psi$
  - $\varphi_c(\psi) = \alpha \int_D \exp(w)\, d\psi$ where $\alpha = \exp\{-E[\theta \log \theta]\}$

Result: f, g linear in w and independent of ψ

- $f(w, \theta) = r(\theta; \alpha) + w$
- $g(w, \theta) = h(\theta; \alpha) + w$
  $\implies$ Can choose r and h instead of f and g

Dynamics (log utility)

- Evolution of individual promised values:

$$w_t\left(w_0, \theta^{t-1}\right) = w_0 + \sum_{i=0}^{t-1} h\left(\theta_i; \alpha\right)$$

- Evolution of period utilities

$$u_t\left(w_0, \theta^{t}\right) = w_0 + \sum_{i=0}^{t-1} h\left(\theta_i; \alpha\right) + r\left(\theta_t\right)$$

$\Longrightarrow$ Cross sectional variance

$$\begin{aligned} Var\left[w_t\left(w_0, \theta^{t-1}\right)\right] &= Var\left(w_0\right) + tVar\left[h\left(\theta; \alpha\right)\right] \\ Var\left[u_t\left(w_0, \theta^{t}\right)\right] &= Var\left(w_0\right) + tVar\left[h\left(\theta; \alpha\right)\right] + Var\left[r\left(\theta\right)\right] \end{aligned}$$

The degree of inequality grows without bound when resources are efficientrly allocated.