Model Uncertainty and Liquidity

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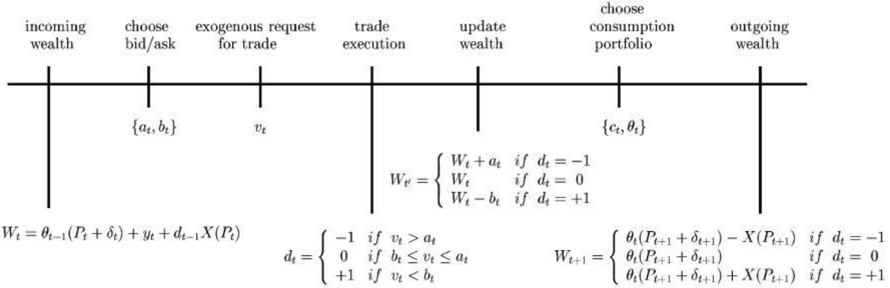
Introduction: Uncertainty and Liquidity

- First common feature of crises: an increase in uncertainty.
- Second common feature of crises: a severe lack of liquidity.
- Can a severe reduction in liquidity result from model uncertainty?

Model: Market Structure

- There is a frictionless market for an underlying asset whose price exogenous is P_t (and exogenous dividend δ_t).
- The market maker sets a bid, b_t, and ask price, a_t, for a derivative whose payoff is $X(P_t)$
  (e.g. one-period call option $X(P_t) = max(P_t - x, 0)$).
- The demand for the derivative is random and characterized by the arrival of "a willingness-to-trade $\widetilde{v}_t$".
  - if $\widetilde{v}_t > a_t$, then a "buy order" is received,
  - if $\widetilde{v}_t < b_t$, then an "ask order" is received,
  - if $b_t \leq \widetilde{v}_t \leq a_t$ no trade takes place.
- Denote the trade outcome $d_t \in \{-1, 0, 1\}$. Let $\Phi(v)$ =Prob($\widetilde{v} < v$). Then,
  - Prob($d_t = -1) = [1 - \Phi(a_t)]$,
  - Prob($d_t = 0) = [\Phi(a_t) - \Phi(b_t)]$,
  - Prob($d_t = 1) = \Phi(b_t)$.

Model: Timeline



$$W_{t'} = \begin{cases} W_t + a_t & if \;\; d_t = -1 \\ W_t & if \;\; d_t = \;\; 0 \\ W_t - b_t & if \;\; d_t = +1 \end{cases}$$

$$W_t = \theta_{t-1}(P_t + \delta_t) + y_t + d_{t-1}X(P_t)$$

$$d_t = \begin{cases} -1 & if \;\; v_t > a_t \\ \;\;0 & if \;\; b_t \le v_t \le a_t \\ +1 & if \;\; v_t < b_t \end{cases}$$

$$W_{t+1} = \begin{cases} \theta_t(P_{t+1} + \delta_{t+1}) - X(P_{t+1}) & if \;\; d_t = -1 \\ \theta_t(P_{t+1} + \delta_{t+1}) & if \;\; d_t = \;\; 0 \\ \theta_t(P_{t+1} + \delta_{t+1}) + X(P_{t+1}) & if \;\; d_t = +1 \end{cases}$$

Model: Preferences and Uncertainty

Let Prob$\{P', \delta', y' | P, \delta, y\} = \pi(P, \delta, y)$

$$U(c_0, \widetilde{c}_1, \widetilde{c}_2, ...) = u(c_0) + \beta \min_{\pi \in \Pi} E_\pi U(\widetilde{c}_1, \widetilde{c}_2, ...)$$

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- If Π is a singleton, then call the market maker a Savage agent.
- Otherwise, call him a Knight agent.

Model: Value Function

$$V(\theta, d, P, \delta, y) =$$

$$\max_{a,b}\{\Phi(b)\{\max_{\theta'}[u\left(\theta(P+\delta)+y+dX(P)-b-\theta' P\right)+\beta\min_{\pi\in\Pi}E_{\pi}\left[V\left(\theta', 1, P', \delta', y'\right)\right]]\}$$

$$+[\Phi(a)-\Phi(b)]\{\max_{\theta'}[u\left(\theta(P+\delta)+y+dX(P)-\theta' P\right)+\beta\min_{\pi\in\Pi}E_{\pi}\left[V\left(\theta', 0, P', \delta', y'\right)\right]]\}$$

$$+[1-\Phi(a)]\{\max_{\theta'}[u\left(\theta(P+\delta)+y+dX(P)+a-\theta' P\right)+\beta\min_{\pi\in\Pi}E_{\pi}\left[V\left(\theta', -1, P', \delta', y'\right)\right]]\}\}$$

Two-Period Model: Portfolio Choice

$$U(d,a,b) = \max_{\theta}\{u(\omega_0 - \theta P_0) + \beta \min_{\pi\in\Pi} E_{\pi}[u(\omega_1 + \theta P_1)]\}$$

where

$$\omega_0 = y_0 + \begin{cases} -b & \text{if} \quad d = 1 \\ 0 & \text{if} \quad d = 0 \\ a & \text{if} \quad d = -1 \end{cases} \quad \text{and } \omega_1 = y_1 + dX(P_1)$$

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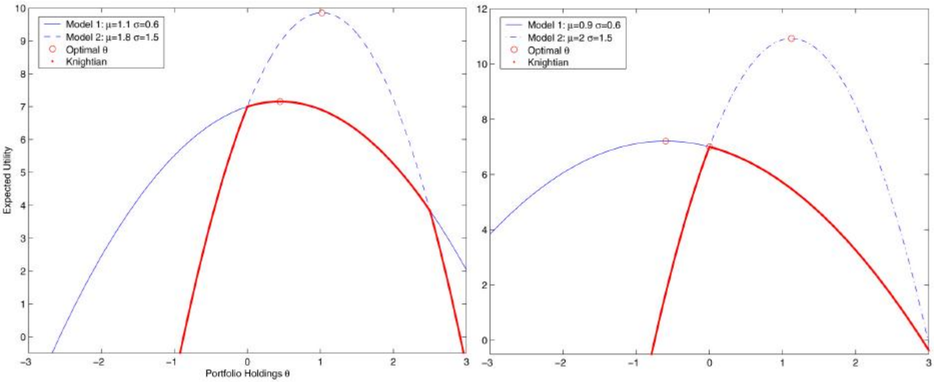
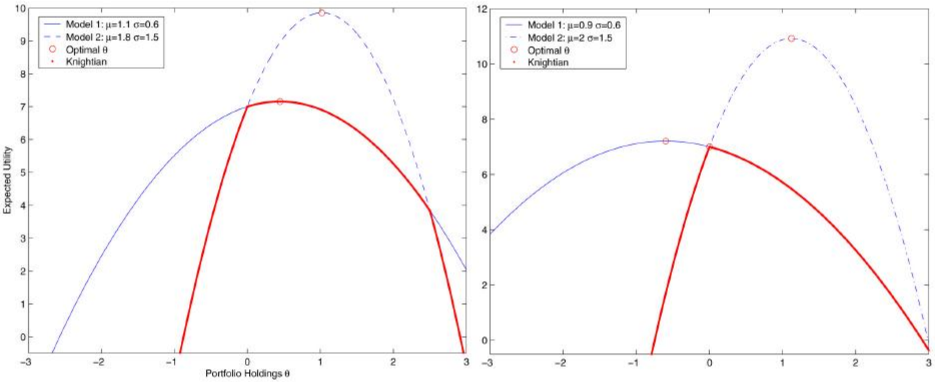
Solve the above problem for a pessimistic Savage agent with beliefs given by $\Pi^1 = \{\pi^1\}$, an optimistic Savage agent with beliefs given by $\Pi^2 = \{\pi^2\}$ and a Knight agent with beliefs given by $\Pi^K = \{\pi^1, \pi^2\}$.

Two-Period Model: Optimal Portfolio Holdings



Two-Period Model: Optimal Portfolio Holdings



$$\theta^K = \theta^1$$

$$-u'\left(\omega_0 - \theta^K P_0\right) P_0 + \beta E_{\pi^1}\left[u'\left(\omega_1 + \theta^K P_1\right) P_1\right] = 0$$

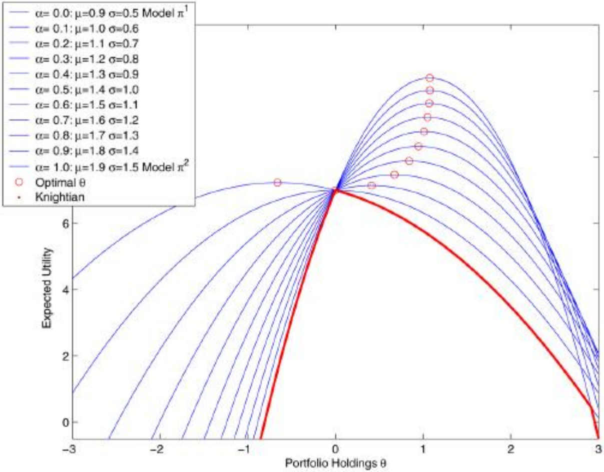
$$0 < \frac{\partial \theta^K}{\partial \omega_0} < \frac{1}{P_0}$$

$$\theta^1 < \theta^K < \theta^2$$

$$E_{\pi^1}\left[u\left(\omega_1 + \theta^K P_1\right)\right] = E_{\pi^2}\left[u\left(\omega_1 + \theta^K P_1\right)\right]$$

$$\frac{\partial \theta^K}{\partial \omega_0} = 0$$

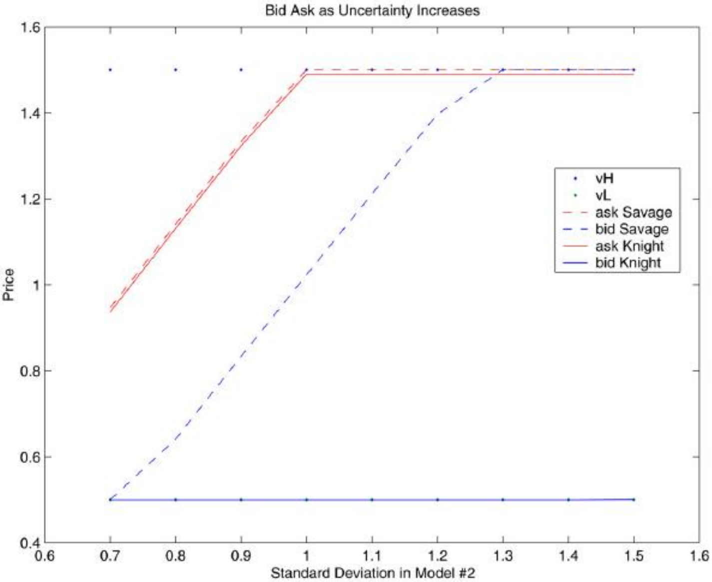
Two-Period Model: Optimal Portfolio Holdings

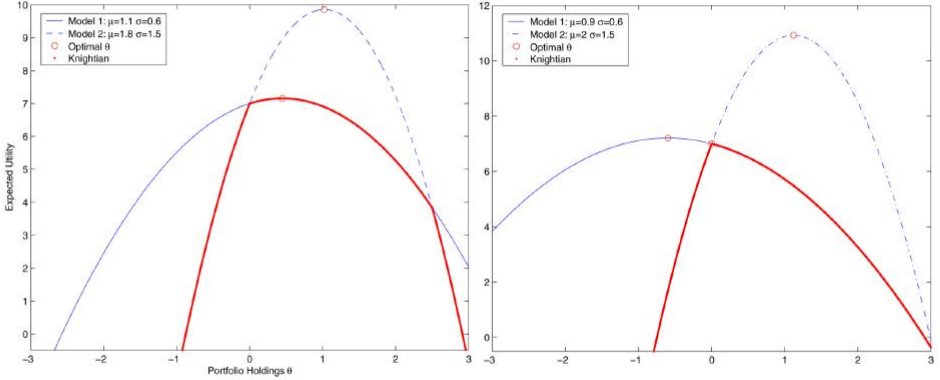


Two-Period Model: The Market-Maker Problem

$$\max_{a,b} \{ [1 - \Phi(a)] U_a + [\Phi(a) - \Phi(b)] U_0 + \Phi(b) U_b \}$$

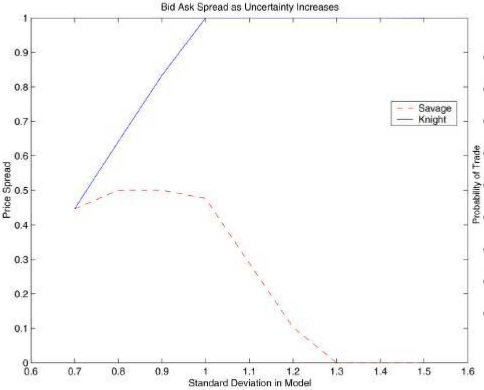
Two-Period Model: The Market-Maker Problem

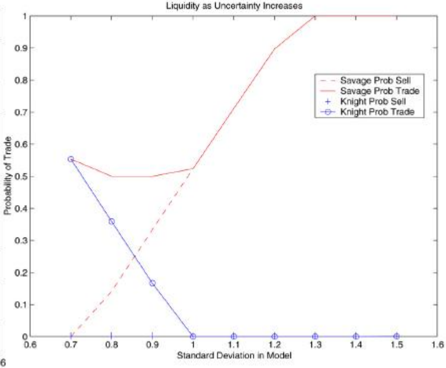




$$
\begin{aligned}
V(\theta, d, P, \delta, y) \quad = \quad & \max_{a,b}\{[1-\Phi(a)]\{\max_{\theta'}[u\left(\theta(P+\delta)+y+dX(P)+a-\theta' P\right)+\beta \min_{\pi\in\Pi} E_\pi\left[V\left(\theta', -1, P', \delta', y'\right)\right]]\} \\
& +[\Phi(a)-\Phi(b)]\{\max_{\theta'}[u\left(\theta(P+\delta)+y+dX(P)-\theta' P\right)+\beta \min_{\pi\in\Pi} E_\pi\left[V\left(\theta', 0, P', \delta', y'\right)\right]]\} \\
& +\Phi(b)\{\max_{\theta'}[u\left(\theta(P+\delta)+y+dX(P)-b-\theta' P\right)+\beta \min_{\pi\in\Pi} E_\pi\left[V\left(\theta', 1, P', \delta', y'\right)\right]]\}\}
\end{aligned}
$$

Two-Period Model: Bid-Ask Spread and Probability of Trade





Infinite Horizon Model

- Four-state Markov process of $(P_t, \delta_t) \in \{(0.75, 0), (0.75, 0.4), (1.25, 0), (1.25, 0.4)\}$.
- Assume there are three market makers with $\Pi^1 = \{\pi^1\}$, $\Pi^2 = \{\pi^2\}$, $\Pi^K = \{\pi^1, \pi^2\}$.
- Other parameters in the example are: utility is logarithmic, exogenous income is constant $y_t = 12.750$, and $\beta = 0.8$.

Infinite Horizon Model