Model Uncertainty and Liquidity

Bryan R. Routledge and Stanley E. Zin

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Presented by Anna Orlik

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Introduction: Uncertainty and Liquidity

- First common feature of crises: an increase in uncertainty.
- Second common feature of crises: a severe lack of liquidity.
- Can a severe reduction in liquidity result from model uncertainty?
Model: Market Structure

- There is a frictionless market for an underlying asset whose price exogenous is $P_t$ (and exogenous dividend $\delta_t$).
- The market maker sets a bid, $b_t$, and ask price, $a_t$, for a derivative whose payoff is $X(P_t)$ (e.g. one-period call option $X(P_t) = \max(P_t - x, 0)$).

- The demand for the derivative is random and characterized by the arrival of "a willingness-to-trade $\tilde{v}_t$".
  - if $\tilde{v}_t > a_t$, then a "buy order" is received,
  - if $\tilde{v}_t < b_t$, then an "ask order" is received,
  - if $b_t \leq \tilde{v}_t \leq a_t$ no trade takes place.

- Denote the trade outcome $d_t \in \{-1, 0, 1\}$. Let $\Phi (v) = \text{Prob}(\tilde{v} < v)$. Then,
  - $\text{Prob}(d_t = -1) = [1 - \Phi (a_t)]$,
  - $\text{Prob}(d_t = 0) = [\Phi (a_t) - \Phi (b_t)]$,
  - $\text{Prob}(d_t = 1) = \Phi (b_t)$.
Model: Timeline

\[ W_t = \theta_{t-1}(P_t + \delta_t) + y_t + d_{t-1}X(P_t) \]

\[ d_t = \begin{cases} 
-1 & \text{if } v_t > a_t \\
0 & \text{if } b_t \leq v_t \leq a_t \\
+1 & \text{if } v_t < b_t 
\end{cases} \]

\[ W_{t+1} = \begin{cases} 
\theta_t(P_{t+1} + \delta_{t+1}) - X(P_{t+1}) & \text{if } d_t = -1 \\
\theta_t(P_{t+1} + \delta_{t+1}) & \text{if } d_t = 0 \\
\theta_t(P_{t+1} + \delta_{t+1}) + X(P_{t+1}) & \text{if } d_t = +1 
\end{cases} \]
Model: Preferences and Uncertainty

Let \( \text{Prob}\{P', \delta', y'|P, \delta, y\} = \pi(P, \delta, y) \)

\[
U(c_0, \tilde{c}_1, \tilde{c}_2, \ldots) = u(c_0) + \beta \min_{\pi \in \Pi} E_{\pi} U(\tilde{c}_1, \tilde{c}_2, \ldots)
\]
Model: Preferences and Uncertainty

Let $\text{Prob}\{P', \delta', y'|P, \delta, y\} = \pi(P, \delta, y)$

$$U(c_0, \tilde{c}_1, \tilde{c}_2, ...) = u(c_0) + \beta \min_{\pi \in \Pi} E_{\pi} U(\tilde{c}_1, \tilde{c}_2, ...)$$

- If $\Pi$ is a singleton, then call the market maker a Savage agent.
- Otherwise, call him a Knight agent.
Model: Value Function

\[ V(\theta, d, P, \delta, y) = \]

\[
\max_{a,b}\{\Phi(b) \{\max_{\theta'}[u(\theta(P + \delta) + y + dX(P) - b - \theta'P) + \beta\min_{\pi \in \Pi}E_{\pi}[V(\theta', 1, P', \delta', y')]]\}
\]

\[ + [\Phi(a) - \Phi(b)]\{\max_{\theta'}[u(\theta(P + \delta) + y + dX(P) - \theta'P) + \beta\min_{\pi \in \Pi}E_{\pi}[V(\theta', 0, P', \delta', y')]]\}
\]

\[ + [1 - \Phi(a)]\{\max_{\theta'}[u(\theta(P + \delta) + y + dX(P) + a - \theta'P) + \beta\min_{\pi \in \Pi}E_{\pi}[V(\theta', -1, P', \delta', y')]]\}\} \]
Two-Period Model: Portfolio Choice

\[ U(d, a, b) = \max_{\theta} \{ u(\omega_0 - \theta P_0) + \beta \min_{\pi \in \Pi} E_\pi \left[ u(\omega_1 + \theta P_1) \right] \} \]

where

\[
\omega_0 = y_0 + \begin{cases} 
-b & \text{if} & d = 1 \\
0 & \text{if} & d = 0 \\
a & \text{if} & d = -1
\end{cases}
\]

and \( \omega_1 = y_1 + dX(P_1) \).
Two-Period Model: Portfolio Choice

\[ U(d, a, b) = \max_{\theta} \{ u(\omega_0 - \theta P_0) + \beta \min_{\pi \in \Pi} \mathbb{E}_\pi [u(\omega_1 + \theta P_1)] \} \]

where

\[ \omega_0 = y_0 + \begin{cases} 
- b & \text{if } d = 1 \\
0 & \text{if } d = 0 \\
- a & \text{if } d = -1 
\end{cases} \]

and \( \omega_1 = y_1 + dX(P_1) \)

Solve the above problem for a pessimistic Savage agent with beliefs given by \( \Pi^1 = \{ \pi^1 \} \), an optimistic Savage agent with beliefs given by \( \Pi^2 = \{ \pi^2 \} \) and a Knight agent with beliefs given by \( \Pi^K = \{ \pi^1, \pi^2 \} \).
Two-Period Model: Optimal Portfolio Holdings

\[ \theta_{K} = \theta_{1} - u' (\omega_{0} - \theta_{K} P_{0}) P_{0} + \beta E \pi_{1} \left[ u' (\omega_{1} + \theta_{K} P_{1}) P_{1} \right] = 0 \]

\[ 0 < \frac{\partial \theta_{K}}{\partial \omega_{0}} < 1 \]

\[ \theta_{1} < \theta_{K} < \theta_{2} \]

\[ E \pi_{2} \left[ u' (\omega_{1} + \theta_{K} P_{1}) \right] = \frac{\partial \theta_{K}}{\partial \omega_{0}} = 0 \]
Two-Period Model: Optimal Portfolio Holdings

\[ \theta^K = \theta^1 \]

\[ -u'(\omega_0 - \theta^K P_0) P_0 + \beta E_{\pi_1} \left[ u' \left( \omega_1 + \theta^K P_1 \right) P_1 \right] = 0 \]

\[ 0 < \frac{\partial \theta^K}{\partial \omega_0} < \frac{1}{P_0} \]

\[ \theta^1 < \theta^K < \theta^2 \]

\[ E_{\pi_1} \left[ u \left( \omega_1 + \theta^K P_1 \right) \right] = E_{\pi_2} \left[ u \left( \omega_1 + \theta^K P_1 \right) \right] \]

\[ \frac{\partial \theta^K}{\partial \omega_0} = 0 \]
Two-Period Model: Optimal Portfolio Holdings

Model Uncertainty and Liquidity
Two-Period Model: The Market-Maker Problem

\[ \max_{a,b} \{ [1 - \Phi(a)]U_a + [\Phi(a) - \Phi(b)]U_0 + \Phi(b)U_b \} \]
Two-Period Model: The Market-Maker Problem

Bid Ask as Uncertainty Increases

- $v_H$
- $v_L$
- Ask Savage
- Bid Savage
- Ask Knight
- Bid Knight

Model Uncertainty and Liquidity
\[ V(\theta, d, P, \delta, y) = \max_{a,b} \left\{ [1 - \Phi(a)] \{ \max_{\theta'} [u(\theta(P + \delta) + y + dX(P) + a - \theta'P) + \beta \min_{\pi \in \Pi} E_{\pi} \left[ V(\theta', -1, P', \delta', y') \right]] \} + [\Phi(a) - \Phi(b)] \{ \max_{\theta'} [u(\theta(P + \delta) + y + dX(P) - \theta'P) + \beta \min_{\pi \in \Pi} E_{\pi} \left[ V(\theta', 0, P', \delta', y') \right]] \} + \Phi(b) \{ \max_{\theta'} [u(\theta(P + \delta) + y + dX(P) - b - \theta'P) + \beta \min_{\pi \in \Pi} E_{\pi} \left[ V(\theta', 1, P', \delta', y') \right]] \} \} \]
Two-Period Model: Bid-Ask Spread and Probability of Trade

Model Uncertainty and Liquidity
Infinite Horizon Model

- Four-state Markov process of $(P_t, \delta_t) \in \{(0.75, 0), (0.75, 0.4), (1.25, 0), (1.25, 0.4)\}$.

- Assume there are three market makers with $\Pi^1 = \{\pi^1\}$, $\Pi^2 = \{\pi^2\}$, $\Pi^K = \{\pi^1, \pi^2\}$.

- Other parameters in the example are: utility is logarithmic, exogenous income is constant $y_t = 12.750$, and $\beta = 0.8$. 
Infinite Horizon Model

Sample Path of Prob. Trade – under model 1

Sample Path of Prob. Trade – under model 2