“Sequential Equilibria in a Ramsey Tax Model”, by Phelan and Stacchetti (E’metrica ’01)

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Setup and Main Idea of the Paper

- This paper studies a Ramsey tax model wo/ gov.'s commitment and:
  - Presents a full recursive characterization of the eqm. set.
  - Develops a dynamic programming method for dynamic policy games between gov. and HH (continuum).
  - Shows that in this setting: Optimal tax of capital in steady state = 0 might not be possible.
- Setting: Dynamic policy game
  - Gov: Chooses $(\tau_{l,t}, \tau_{k,t})_t$ s.t. $G_t = \tau_{k,t} p_{k,t} K_t + \tau_{l,t} p_{l,t} L_t$.
  - HH: Chooses $(l_t, c_t, \theta_t)_t$ according to $U \equiv \sum_t E[\beta^t (u(l_t, c_t) + g(G_t))]$ given taxes, prices and

$$y_t = (1 - \tau_{k,t}) p_{k,t} K_t + (1 - \tau_{l,t}) p_{l,t} l_t$$
$$k_{t+1} = \theta_t y_t \text{ } and \text{ } c_t = (1 - \theta_t) y_t.$$

  - **Assumptions:** Standard for $u(l, c)$ and firms; $u_c(l, 0) \leq M$ and full depreciation.

Literature Review

- Kydland & Prescott ('77): Point out problems of characterizing recursively the Ramsey problem w/ commitment:
  - Time 0 solution is time inconsistent.
  - Key Insight: Current HH's optimal decision depend on the "state" *and* on beliefs of future gov. policies which must be validated.
- Kydland & Prescott ('80): Solve this problem by augmenting the "state" and adding a (co)state variable that tracks "promised values".
  - Include Lagrange Multiplier of the HH or future marginal utility.
  - State space of this (co)state variable is also determined recursively.
- Chari and Kehoe ('90): Model lack of commitment as a policy game between Gov and HH:
  - Characterize set of "sustainable" eqm. by using threats of reversion to "bad" equilibria.
  - Ramsey Eqm is sustainable if β is low enough.
  - *Assume* that "bad" eqm exists and it is known.
- Marcet and Marimon ('94 '98): Propose an optimal control approach.

Extension of APS to *dynamic* policy games

- APS ('86): Apply dynamic programming to *repeated* games:
  - Keep track of continuation values as a (co)state.
  - Characterize all values that are associated w/ sustainable eqm.
- This paper: Characterize the eqm values recursively of sustainable eqm in a *dynamic* game by:
  - keeping track of *future* marginal value of capital.
  - Making sure that HH beliefs are manipulated so as to ensure the worst possible "credible" payoff for the government in case he wants to deviate.

  - APS ('86) but for a *dynamic* game.
  - CK ('90) but without a priori knowledge of "bad eqm".
  - MM ('98 '94) but game theoretic approach.

The Dynamic Policy Game

- $\Gamma(k)$: Dynamic game between the gov. and the HH w/ initial capital k.
- x_t (public) iid-random outcome.
- $h^t = (\tau^t, x^t)$ (public) history at time t.
- $\tau_t = \sigma_G(t)(h^{t-1}, x_t)$, where $\tau_t = (\tau_{l,t}, \tau_{k,t})$.
- $a_t = \sigma_C(t)(h^{t-1}, x_t, \tau_t)$ where $a_t = (l_t, \theta_t)$.
- $\Phi_G(k_0, \sigma) = (1-\beta)U$ (gov is benevolent).
- **Def.** A C.E. *given* $\tau = (\tau_t)_t$ $(CE(\Gamma(k|\tau)))$ is $q \equiv (l_t, c_t, k_{t+1})(x^t))_t$ w/

$$p_{k,t}(x^t) = (1 - \tau_{k,t}(x^t))f_k(k_t(x^{t-1}), l_t(x^t))$$
$$p_{l,t}(x^t) = (1 - \tau_{l,t}(x^t))f_l(k_t(x^{t-1}), l_t(x^t)) \textit{ and}$$
$$m_{t+1}(x^t) \equiv E\left[p_{k,t+1}(x^{t+1})U_c(l_{t+1}(x^{t+1}), c_{t+1}(x^{t+1}))|x_t\right].$$

The Dynamic Policy Game (cont.)

- **Def.** A $\sigma = (\sigma_C(t), \sigma_G(t))_t$ is a symmetric seq. eqm (SSE) for $\Gamma(k_0)$ if $\forall t,\ h^{t-1},\ k_t$ *and* $x_t \in [0,1]$:
  - (i) $\Phi_G(k_t, \sigma|_{(h^{t-1},x_t)}) \geq \Phi_G(k_t, (\sigma_C|_{(h^{t-1},x_t)}, \gamma|_{x_t}))$
  - (ii) $\forall \hat{\tau}_0 \in [0,1]^2$, let $(\hat{\tau}, \hat{q})$ be generated by $\hat{\tau}_0$ and $\sigma|_{(h^{t-1},x_t,\hat{\tau}_0)}$ $\Rightarrow \hat{q} \in CE(\Gamma(k_t|\hat{\tau}))$.
- That is, $\sigma \in SSE(\Gamma(k_0))$ provided that the gov. and HH. are playing best responses (to each other).
- The goal is to characterize this recursively.

Equilibrium Value Correspondence

- A sequence of lifetime payoffs is produced by $SSE(\Gamma(k))$ only if it is the sum (or weighted average) of: current eqm payoff + lifetime payoffs of continuation of SSE.
- So we need (APS insight)

$$V(k) = \{\text{set of functions } \hat{v}(\cdot) \text{ that can be produced by } SSE(\Gamma(k))\}$$

- In this setting it simplifies to $V(k) \subseteq \mathbb{R}^2$,

$$V(k) = \{\Phi(k,\sigma) = (\Phi_C(k,\sigma), \Phi_G(k,\sigma)) : \sigma \in SSE(\Gamma(k))\}$$

- That is, we need (a) future after tax marginal value of capital ($\Phi_C(k,\sigma)$) and (b) gov. lifetime pay-offs ($\Phi_G(k,\sigma)$).
- $V(k)$ is the correspondence we need to characterize recursively using the "APS operator".

Why m?

- m is what we need to keep track of HH optimal behavior (i.e., $CE(\Gamma(k|\tau))$).
- APS insight: In order to construct the sequence $q \in CE(\Gamma(k|\tau))$, we have to "piece together" results of *static* games w/ appropriately modified payoffs.
- **Thm.** We say that $q_t \in CE(k_t, \tau_t, m_{t+1})$ if q_t is allocation of eqm for the *static* economy with payoffs: $u(l_t, c_t) + \beta m_{t+1} k_{t+1}$.
  Under some regularity conditions
  $q \in CE((\Gamma(k|\tau)) \iff q_t \in CE(k_t, \tau_t, m_{t+1}), \ \forall t$ and m_{t+1} is given by

$$m_{t+1}(x^t) \equiv E\left[p_{k,t+1}(x^{t+1}) U_c(l_{t+1}(x^{t+1}), c_{t+1}(x^{t+1})) | x_t\right].$$

Recursive characterization of $SSE(\Gamma(k))$

- $\sigma \in SSE$ generates (I) a current outcome (τ, l, c, k_+) and (II) an eqm continuation profile $(m_+, v_+) = \Phi(k_+, \sigma)$.
- Sufficient conditions for $\sigma \in SSE$ are casted in terms of (I)-(II): **Consistency** and **Admissibility**
- **Consistency** A vector $\zeta = (\tau, l, c, k_+, m_+, v_+)$ is consistent w.r.t. V at k if:
  - (a) $(l, c, k_+) \in CE(k, \tau, m_+)$.
  - (b) $(m_+, v_+) \in V(k_+)$.

Recursive characterization of $SSE(\Gamma(k))$ (cont.)

- **Admissibility** A vector $\zeta = (\tau, l, c, k_+, m_+, v_+)$ is admissible w.r.t. V at k if it is consistent and if $\Psi_G \equiv (1-\beta)[U(l,c)+g(G)] + \beta v_+$,

$$\Psi_G(k,\zeta) \geq \underbrace{\max_{\tau'} \left\{ \underbrace{\min_{(l',c',k'_+,v'_+)} \Psi_G(k,\zeta') : \zeta' \textit{ is consistent } (w.r.t.\ V)}_{(A)} \right\}}_{(B)}$$

- Admissibility is Gov I.C.: If the gov. announces different taxes, the HH beliefs are manipulated to ensure that he will be picking the taxes (B) from the worst "credible" (hence the "is consistent (w.r.t. V)" inside (B)) payoff (A).
- **Remark:** Ψ and Φ convey the same information, but they are different because they are defined in different domains.

Recursive characterization of $SSE(\Gamma(k))$ (cont.)

- Let $(B) \equiv \pi_V(k)$. Define:

$$\hat{B}(W)(k) = \{\Psi(k,\zeta) : \zeta \textit{ is consistent w.r.t. } W\}$$

and

$$B(W)(k) = co(\{(m,v) \in \hat{B}(W)(k) : v \geq \pi_W(k)\})$$

- **APS Fixed point**
  - Let W be a compact and cvx values correspondence, if $W \subset B(W)$ then $B(W) \subset V$.
  - If W is u.h.c. then $B(W)$ u.h.c.
  - Graph(V) is bbd.
  - V is such that (a) $V = B(V)$ and $V \supseteq W : W = B(W)$.
  - $V = \lim W_n$, with $W_{n+1} = B(W_n)$.

Best Equilibria and Steady States

- **Def.** ζ^s is a S.S. if:
  - (C.E.) $(l^s, c^s, k^s) \in CE(k^s, \tau^s, m^s)$.
  - (I.C.) $(m^s, v^s) = \Psi(k^s, \zeta^s)$.
- If a best eqm reaches a S.S. w/ ζ^s then in the S.S. either taxes to capital are zero or the S.S represents the wort eqm in $\Gamma(k^s)$.
- **Thm.** Let $\sigma \in SSE(\Gamma(k_0))$ and $\Phi_G(k_0, \sigma) = \overline{V}(k_0, \Phi_C)$ and "$\sigma \rightarrow \zeta^s$" then (i) $\tau_k^s = 0$ or (ii) $v^s = \pi_V(k^s)$.
- Intuition:
  - If in the limit the I.C. is not binding $\Rightarrow$ Chamley-Judd result is attained.
  - If in the limit the I.C. *is* binding $\Rightarrow$ (a) Value of government must be min possible (i.e., $\pi_V(k^s)$) and (b) $\tau_k^s \neq 0$.
- This depends on (amongst other things) β.

Concluding Remarks

- Illustrate results and algorithm in an example.
- Presents a full characterization of the eqm. set of a Ramsey tax model w/ capital and wo/ gov.'s commitment.
- Develops a dynamic programming method for dynamic games between gov. and HH (continuum).
  - Key Insight: Since HH are homogenous and actions do not affect prices the HH intertemporal incentives can be completely captured by only one number: "tomorrow's" marginal utility of capital.
- Studies steady state of best sustainable eqm wo/ commitment: Optimal tax of capital may not be 0.