Assessing Specification Errors in Stochastic Discount Factor Models

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How can we compare misspecified asset pricing models?

- at date t a set of assets is purchased at price q_t
- these assets have a payoff of $x_{t+\tau}$ at date $t + \tau$
- let $\mathcal{F}$ be the conditioning observation observable at $t + \tau$ and L^2 the space of all random variables with finite second moments in $\mathcal{F}$
- inner product and norm: $\langle h_1 | h_2 \rangle = E(h_1 h_2)$, $\|h\| = \langle h | h \rangle^{1/2}$
- P, the space of portfolio payoffs used in the econometric analysis is a closed linear subspace of L^2. For the empirical analysis, let $P = \{x \cdot c : c \in \mathbb{R}^n\}$ with x being an n dimensional vector with entries in L^2

the *expected* price $\pi(p)$

- π is continuous and linear on P, and there exists a $p \in P$ such that $\pi(p) = 1$

Stochastic Discount Factors

- an admissible sdf is a random variable in L^2 such that

$$\pi(p) = E(pm) \forall p \in P \tag{1}$$

  Let $\mathcal{M}$ be the space of all admissible stochastic discount factors. Further, let $\mathcal{M}^{++}$ be the set of all stochastic discount factors that are positive with probability one and let $\mathcal{M}^{+} = closure(\mathcal{M}^{++})$
- call the extension of π to all $h \in L^2$ $\pi_m(h) = Ehm$. This pricing functional does not induce arbitrage opportunities if $m \in \mathcal{M}^{++}$

- an asset pricing model gives us a proxy for a stochastic discount factor:

$$\pi_a(h) = E(yh) \forall h \in L^2 \tag{2}$$

Least Squares Problems

- Problem 1:

$$\delta = \min_{m \in \mathcal{M}} \|y - m\| \tag{3}$$

- Problem 2:

$$\delta^+ = \min_{m \in \mathcal{M}^+} \|y - m\| \tag{4}$$

Maximum Pricing Errors

- $$\delta = \max_{p \in P, \|p\|=1} |\pi_a(p) - \pi(p)| \tag{5}$$

- $$\delta^+ = \min_{m \in \mathcal{M}^+} \max_{h \in L^2, \|h\|=1} |\pi_m(h) - \pi_a(h)| \tag{6}$$

Duality to solve the LS Problems: Problem 2

- $$(\delta^+)^2 = \min_{m \in L^2, m \geq 0} \|y - m\|^2 \quad (7)$$

  subject to $E(mx) = Eq$

- $$(\delta^+)^2 = \max_{\lambda \in \mathbb{R}^n} \min_{m \in L^2, m \geq 0} (E(y-m)^2 + 2\lambda' E(xm) - 2\lambda' Eq) \quad (8)$$

- solving the inner minimization problem we get:

  $$(\delta^+)^2 = \max_{\lambda \in \mathbb{R}^n} E(y^2 - [(y - \lambda' x)^+]^2 - 2\lambda' q) \quad (9)$$

- this needs to be solved numerically

Duality to solve the LS Problems: Problem 1

- calculations similar to the ones for problem 2 give us:

$$(\delta^+)^2 = \max_{\lambda \in \mathbb{R}^n} E(y^2 - [(y - \lambda' x)]^2 - 2\lambda' q) \tag{10}$$

- FONC:

$$E[x(y - \tilde{\lambda}' x) - q = 0] \tag{11}$$

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$$\tilde{\lambda} = (Exx')^{-1} E(xy - q) \tag{12}$$

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$$\delta = [E(xy - q)'(Exx')^{-1} E(xy - q)]^{1/2} \tag{13}$$

- $$d_T = \{\max_{\lambda \in \mathbb{R}^n} T^{-1} \sum_{t=1}^{T} [y_{t+\tau}^2 - (y_{t+\tau} - \lambda' x_{t+\tau})^2 - 2\lambda' q_t]\}^{1/2} \quad (14)$$

- $$d_T^+ = \{\max_{\lambda \in \mathbb{R}^n} T^{-1} \sum_{t=1}^{T} [y_{t+\tau}^2 - (y_{t+\tau} - \lambda' x_{t+\tau})^{+2} - 2\lambda' q_t]\}^{1/2} \quad (15)$$

Hansen, Heaton and Luttmer (1995)

- $$T^{1/2}(d_T - \delta) \Longrightarrow^D N[0, \widehat{\sigma}^2/(4\delta^2)] \tag{16}$$

- $$T^{1/2} \sum_{t=1}^{T} [y_{t+\tau}^2 - (y_{t+\tau} - \lambda' x_{t+\tau})^{+2} - 2\lambda' q_t - \delta^2] \Longrightarrow^D N[0, \widehat{\sigma}^2] \tag{17}$$

- $$\frac{T^{1/2} d_T}{2 s_T}(d_T - \delta) \Longrightarrow^D N[0, 1] \tag{18}$$

Specification Errors for Power Utility

γ	δ	δ^+	χ^2
	$\beta = 0.95$		
0	0.332	0.333	2,824.9
	(0.054)	(0.054)	
1	0.333	0.333	2,850.9
	(0.053)	(0.054)	
5	0.333	0.333	1,557.7
	(0.053)	(0.054)	
10	0.333	0.333	705.9
	(0.053)	(0.053)	
15	0.334	0.334	419.2
	(0.052)	(0.053)	
	$\beta = 1.00$		
0	0.329	0.329	46.6
	(0.054)	(0.055)	
1	0.329	0.329	42.6
	(0.054)	(0.054)	
5	0.329	0.329	47.8
	(0.054)	(0.054)	
10	0.328	0.328	55.7
	(0.053)	(0.054)	
15	0.328	0.328	58.1
	(0.053)	(0.054)	
Minimized	0.327	0.327	35.2
	(0.053)	(0.053)	

Lagrange Multipliers for Power Utility

	$\beta = 1.00$		
Security	$\gamma = 0$	$\gamma = 5$	$\gamma = 15$
1	8.60	8.61	8.63
	(3.89)	(3.88)	(3.88)
2	1.81	1.80	1.77
	(0.96)	(0.97)	(0.98)
3	4.74	4.79	4.89
	(1.80)	(1.81)	(1.82)
4	−13.46	−13.47	−13.50
	(3.71)	(3.72)	(3.72)
5	3.76	3.75	3.73
	(0.89)	(0.88)	(0.88)
6	−5.56	−5.58	−5.64
	(1.77)	(1.77)	(1.78)

Specification Errors for Habits

γ	δ	δ^+	χ^2
	$\beta = 0.95$		
0	0.332	0.332	2,762.1
	(0.054)	(0.054)	
1	0.332	0.332	2,837.7
	(0.054)	(0.054)	
5	0.332	0.332	807.6
	(0.053)	(0.054)	
10	0.332	0.332	244.0
	(0.053)	(0.053)	
15	0.332	0.332	123.9
	(0.052)	(0.053)	
	$\beta = 1.00$		
0	0.328	0.339	42.2
	(0.054)	(0.055)	
1	0.328	0.328	42.3
	(0.054)	(0.055)	
5	0.328	0.328	42.1
	(0.054)	(0.055)	
10	0.328	0.329	42.8
	(0.053)	(0.054)	
15	0.328	0.329	42.8
	(0.053)	(0.053)	
Minimized	0.328	0.328	27.5
	(0.053)	(0.054)	

Specification Errors for Linear Factor Models

Variable Factor	δ	χ^2
Coefficients Estimated by Minimizing the Specification Error		
Equally-weighted return	0.286	30.1
	(0.054)	
Value-weighted return	0.289	33.5
	(0.052)	
Consumption growth	0.325	38.9
	(0.052)	
Coefficients Estimated by Minimizing the Value of the χ^2		
Equally-weighted return	0.286	29.3
Value-weighted return	0.290	31.5
Consumption growth	1.57	12.0