Rational Expectations in Games

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Main Idea

- Standard "recommendation of game theory": play some specified strategy *known to all players* (NE).
- But in the real-life even if the game is commonly known, the *context* (i.e. the attitudes of the players, their expectations about each other, custom, history, etc...) usually is not.
  (Roth et al. (1991))
- What can we say about "*game situations*"? What is the appropriate solution concept? "Recommendation"?

The Game Situation

Game situation Γ: (a) a game G as defined by its strategy sets and payoff functions, and (b) a belief system.

- The Game G
  - n Players indexed by i.
  - Strategy space $S = S_1 \times S_2 \times ... \times S_n$.
  - Payoff functions $h_1, h_2, ..., h_n$ such that $h_i : S \rightarrow \mathcal{R}$ for $i = 1, 2, ..., n$.
- Belief system B for G
  - For each player i a finite set T_i whose members t_i are called types of i;
  - For each type t_i of each player i:
    - a strategy $s_i(t_i)$, and
    - a probability distribution on (n-1)-tuples of types of the other players, called t_i's theory: $\pi_i\left(t^{-i} \mid t_i\right)$.

Examples: The Game Situation

Payoffs

	L	R
T	$6, 6$	$2, 7$
B	$7, 2$	$0, 0$

Matt's beliefs

	L	R
T	$1/2$	$1/2$
B	$7/8$	$1/8$

Colin's beliefs

	L	R
T	$1/2$	$7/8$
B	$1/2$	$1/8$

$i = \{M, C\}$

$T_M = \{T, B\}, T_C = \{L, R\}$

Theory of $t_M = B$ is given by $\pi_M(L \mid B) = 7/8$ and $\pi_M(R \mid B) = 1/8$.

Expectation of $t_M = B$ equals $6\frac{1}{8}$.

Expectation of $t_C = R$ equals $6\frac{1}{8}$.

However, these expectations are mutually inconsistent: $(6\frac{1}{8}, 6\frac{1}{8})$ is not a feasible payoff.

Definitions

- A type of a player is **rational** if the strategy it prescribes maximizes his expected payoff given its theory.
- **Common knowledge of rationality (CKR)**: all types of all players are rational, and all know that, and all players know that all other players know that all players are rational, etc...
- **Common prior (CP)**: is a probability distribution π on $T_1 \times T_2 \times ... \times T_n$ such that $\pi_i \left(t^{-i} \mid t_i \right) = \pi \left(t \right) / \pi \left(t_i \right)$
- A **belief system is rational** if both CKR and CP obtain.
- A **rational expectation (RE)** in G is an expected payoff of some type of the player in some rational system for G.
- **Value of the game**: as in von Neumann's Minimax Theorem of 1928

Subjective Beliefs and Common Prior

Payoffs

	L	R
T	6, 6	2, 7
B	7, 2	0, 0

Matt's beliefs

	L	R
T	1/2	1/2
B	7/8	1/8

Colin's beliefs

	L	R
T	1/2	7/8
B	1/2	1/8

Common prior

	L	R
T	7/22	7/22
B	7/22	1/22

$\pi_i \left(t^{-i} \mid t_i \right) = \pi(t) / \pi(t_i)$

$\pi_M (L \mid B) = (7/22) / (7/22 + 1/22) = 7/8$

$\pi_C (B \mid L) = (7/22) / (7/22 + 7/22) = 1/2$

$\pi_M (R \mid B) = (1/22) / (7/22 + 1/22) = 1/8$

Correlated Equilibrium (Aumann (1974))

A **correlated equilibrium** of G is a probability distribution ρ on pure strategy profiles, such that if "chance/nature" chooses a strategy profile in accordance with ρ , and informs each player only of *his* strategy in the chosen profile, then it is optimal for *him* to play that strategy, assuming that the others play their strategies.

Correlated Equilibrium in Game G

Payoffs

	L	R
T	$6, 6$	$2, 7$
B	$7, 2$	$0, 0$

Nash equilibria: $(7, 2)$, $(2, 7)$, and a mixed one yielding $\left(4\frac{2}{3}, 4\frac{2}{3}\right)$.

Now let $\rho(T, L) = \rho(B, L) = \rho(T, R) = \frac{1}{3}$.

Suppose player I is assigned B. Then, he would not want to deviate supposing the other player played their assigned strategy since he will get 7.

In turn, suppose player I is assigned T. Then, the other player will play L or R with probability $\frac{1}{2}$ and $\frac{1}{2}$, respectively. Hence, playing T as instructed, yields the expected payoff of $6(1/2) + 2(1/2) = 4$, while the expected payoff of deviating from nature's instruction and playing B equals $0(1/2) + 7(1/2) = 3.5$.

In both cases player I obeys the instructions of nature.
Symmetrically, so will player II.

Consequently, ρ is a correlated equilibrium of game G. Interestingly, the expected payoff for this equilibrium is $7(1/3) + 2(1/3) + 6(1/3) = 5$.

RE vs. CE

Proposition (Aumann (1974)) The expected payoff to every correlated equilibrium in a zero-sum game is the value of the game.

But is every RE a CE?

Theorem B. The rational expectations in a game G are precisely the payoffs to correlated equilibria in the doubled game $2G$.

Payoffs

	L_1	L_2	R_1	R_2
T_1	$6, 6$	$6, 6$	$2, 7$	$2, 7$
T_2	$6, 6$	$6, 6$	$2, 7$	$2, 7$
B_1	$7, 2$	$7, 2$	$0, 0$	$0, 0$
B_2	$7, 2$	$7, 2$	$0, 0$	$0, 0$

On the Way to the Main Result

(1) Every RE in G is a CE in $2G$, and every CE in $2G$ is a RE in G (by Theorem B)
(2) Every CE in G is a CE in $2G$ (multiply by 0 the duplications)
(3) Every CE in G is a RE in G (by (1) and (2))

(4) Every RE is at least the player's maximin payoff v (by Prop. C(iii))
(5) Every CE is at least the player's maximin payoff v (by (3) and (4))

(6) If all CE payoffs are the maximin payoffs v then v is the only RE. (by Prop. E)
(7) The expected payoff to every CE is equal v. (Aumann 1974)

(8) Theorem A (by (6) and (7)).

Main Result

Theorem A. The expectation of any two-person zero-sum game situation with common knowledge of rationality and common priors is the value of the underlying game.

$\Rightarrow$ Rational expectations are not context-specific (in the above types of game situations).

CKR and CP: one without the other

(a)

Payoffs

	L	R
T	$1, -1$	$-1, 1$
B	$-1, 1$	$1, -1$

Matt's beliefs

	L	R
T	0.9	0.1
B	0.1	0.9

Colin's beliefs

	L	R
T	0.1	0.9
B	0.9	0.1

(b)

The common prior

	L_1	L_2	R_1	R_2	L_3
T_1	0.1	0.1	0	0	0
B_1	0.1	0	0.1	0	0
B_2	0	0.1	0	0.1	0
T_2	0	0	0.1	0	0.1
T_3	0	0	0	0.1	0.1