When is Market Incompleteness Irrelevant for the Price of Aggregate Risk (and when is it not)?

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Main Idea

- Intriguing question posed in the title.
- Suggested answer: e.g. Constantinides and Duffie (1996).
- Here: general conditions under which closing down insurance markets for idiosyncratic risk does not increase the risk premium that stocks command over bonds.

Quartet of Models Studied

Table 1: Summary of Four Models

Model	*Idiosyncr. Shocks*	*Aggregate Shocks*	*Arrow Securities*
Bond	Yes	Yes	No
Arrow	Yes	Yes	Yes
BL	No	Yes	Yes
Bewley	Yes	No	N/A

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1st step: $\left.\begin{array}{c}\text{Bewley} \Leftrightarrow \text{Arrow} \\ \text{Bewley} \Leftrightarrow \text{Bond}\end{array}\right\} \Rightarrow \text{Arrow} \Leftrightarrow \text{Bond}$

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2nd step: $\left.\begin{array}{c}\text{BL} \Leftrightarrow \text{Bond} \\ \text{BL} \Leftrightarrow \text{Arrow}\end{array}\right\} \Rightarrow$the same risk premia in complete and incomplete markets (subject to certain assumptions).

Basic Framework

- Continuum of individuals of measure 1.
- Uncertainty: $s^t = (y^t, z^t) = \left(s^{t-1}, y_t, z_t\right)$
- Preferences:

$$U(c)(s_0) = \sum_{t=0}^{\infty} \sum_{s^t \geq s_0} \beta^t \pi\left(s^t | s_0\right) \frac{c_t\left(s^t\right)^{1-\gamma}}{1-\gamma}$$

$$U(c)\left(s^t\right) = u\left(c_t\left(s^t\right)\right) + \beta \sum_{s_{t+1}} \pi\left(s_{t+1} | s_t\right) U(c)\left(s^t, s_{t+1}\right)$$

- Endowments:
  - aggregate: $e_t\left(z^t\right)$, $\lambda\left(z^{t+1}\right) = \frac{e_{t+1}\left(z^{t+1}\right)}{e_t\left(z^t\right)}$
  - labor income: $\eta_t\left(s^t\right) = \eta\left(y_t, z_t\right) e_t\left(z^t\right)$
  - capital income: $\alpha e_t\left(z^t\right)$
  - initial wealth: θ_0, Θ_0

Assumptions

1. Aggregate endowment growth is a function of the current aggregate shock only, i.e. $\lambda\left(z^{t+1}\right) = \lambda\left(z_{t+1}\right)$.

2. Individual endowment shares are $\eta\left(y_t, z_t\right)$ are functions of the current idiosyncratic state, y_t, only, i.e. $\eta\left(y_t, z_t\right) = \eta\left(y_t\right)$.

3. Idiosyncratic shocks are independent of aggregate shocks. Hence, transition probabilities can be decomposed as
$\pi\left(z_{t+1}, y_{t+1} | z_t, y_t\right) = \varphi\left(y_{t+1} | y_t\right) \phi\left(z_{t+1} | z_t\right)$.

4. Aggregate endowment growth is i.i.d., i.e. $\phi\left(z_{t+1} | z_t\right) = \phi\left(z_{t+1}\right)$

Transformation of the Growth Model into a Stationary Model

- Growth-deflated consumption allocations: $\widehat{c}_t\left(s^t\right)=\frac{c_t(s^t)}{e_t(z^t)}$ for all s^t.
- Growth-adjusted probabilities: $\widehat{\pi}\left(s_{t+1}|s_t\right)=\frac{\pi(s_{t+1}|s_t)\lambda\left(z^{t+1}\right)^{1-\gamma}}{\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\lambda(z^{t+1})^{1-\gamma}}$.
- Growth-adjusted discount factor: $\widehat{\beta}\left(s^t\right)=\beta\sum_{s_{t+1}}\pi\left(s_{t+1}|s_t\right)\lambda\left(z^{t+1}\right)^{1-\gamma}$.
- Detrended lifetime expected continuation utility

$$\widehat{U}\left(\widehat{c}\right)\left(s^t\right)=u\left(\widehat{c}_t\left(s^t\right)\right)+\widehat{\beta}\left(s^t\right)\sum_{s_{t+1}}\widehat{\pi}\left(s_{t+1}|s_t\right)\widehat{U}\left(\widehat{c}\right)\left(s^t,s_{t+1}\right)\quad(*)$$

Proposition. $U\left(c\right)\left(s^t\right)\geq U\left(c'\right)\left(s^t\right)\Rightarrow\widehat{U}\left(\widehat{c}\right)\left(s^t\right)\geq\widehat{U}\left(\widehat{c}'\right)\left(s^t\right)$, $\forall s^t,\forall c,c'$

Bewley Model

Each household chooses consumption, $\{\widehat{c}_t(\theta_0, y^t)\}$, bond holdings, $\{\widehat{a}_t(\theta_0, y^t)\}$, and share holdings, $\{\widehat{\sigma}_t(\theta_0, y^t)\}$, to maximize lifetime utility, as in eq. $(*)$, subject to a budget constraint

$$\widehat{c}_t\left(y^t\right) + \frac{\widehat{a}_t\left(y^t\right)}{\widehat{R}_t} + \widehat{\sigma}_t\left(y^t\right)\widehat{v}_t = \eta\left(y_t\right) + \widehat{a}_{t-1}\left(y^{t-1}\right) + \widehat{\sigma}_{t-1}\left(y^{t-1}\right)\left(\widehat{v}_t + \alpha\right)$$

and one of the two borrowing constraints

$$\begin{aligned} \frac{\widehat{a}_t\left(y^t\right)}{\widehat{R}_t} + \widehat{\sigma}_t\left(y^t\right)\widehat{v}_t &\geq \widehat{K}_t\left(y^t\right) \quad \forall y^t \\ \widehat{a}_t\left(y^t\right) + \widehat{\sigma}_t\left(y^t\right)\left(\widehat{v}_{t+1} + \alpha\right) &\geq \widehat{M}_t\left(y^t\right) \quad \forall y^t \end{aligned}$$

Equilibrium result: $\widehat{R}_t = \frac{\widehat{v}_{t+1} + \alpha}{\widehat{v}_t}$.

Arrow Model

Each household chooses consumption, $\{\widehat{c}_t(\theta_0, s^t)\}$, trading strategies for Arrow securities, $\{\widehat{a}_t(\theta_0, s^t, z_{t+1})\}$, and share holdings, $\{\sigma_t(\theta_0, s^t)\}$, to maximize lifetime utility, as in eq. $(*)$, subject to a budget constraint

$$\begin{aligned} & \widehat{c}_t\left(s^t\right) + \sum_{z_{t+1}} \widehat{a}_t\left(s^t, z_{t+1}\right) \widehat{q}_t\left(z^t, z_{t+1}\right) + \sigma_t\left(s^t\right) \widehat{v}_t\left(z^t\right) \\ \leq \quad & \eta\left(y_t\right) + \widehat{a}_{t-1}\left(s^{t-1}, z_t\right) + \sigma_{t-1}\left(s^{t-1}\right)\left[\widehat{v}_t\left(z^t\right) + \alpha\right] \end{aligned}$$

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$$\begin{aligned} \sum_{z_{t+1}} \widehat{a}_t\left(s^t, z_{t+1}\right) \widehat{q}_t\left(z^t, z_{t+1}\right) + \sigma_t\left(s^t\right) \widehat{v}_t\left(z^t\right) & \geq \widehat{K}_t\left(y^t\right) \\ \widehat{a}_t\left(s^t, z_{t+1}\right) + \sigma_t\left(s^t\right)\left[\widehat{v}_{t+1}\left(z_{t+1}\right) + \alpha\right] & \geq \widehat{M}_t\left(y^t\right) \quad \forall z_{t+1} \end{aligned}$$

Equivalence Result I.

Theorem The equilibrium of the stationary Bewley model can be mapped into equilibrium of the Arrow model with growth with

$$
\begin{aligned}
c_t\left(\theta_0, s^t\right) &= \widehat{c}_t\left(\theta_0, y^t\right) e_t\left(z^t\right) \\
\sigma_t\left(\theta_0, s^t\right) &= \widehat{\sigma}_t\left(\theta_0, y^t\right) \\
a_t\left(\theta_0, s^t, z_{t+1}\right) &= \widehat{a}_t\left(\theta_0, y^t\right) e_t\left(z^{t+1}\right) \\
v_t\left(z^t\right) &= \widehat{v}_t e_t\left(z^t\right) \\
q_t\left(z^t, z_{t+1}\right) &= \frac{1}{\widehat{R}_t}\frac{\widehat{\phi}\left(z_{t+1}\right)}{\lambda\left(z_{t+1}\right)} = \frac{1}{\widehat{R}_t}\frac{\phi\left(z_{t+1}\right)\lambda\left(z_{t+1}\right)^{-\gamma}}{\sum_{z_{t+1}}\phi\left(z_{t+1}\right)\lambda\left(z_{t+1}\right)^{1-\gamma}}
\end{aligned}
$$

Bond Model

Each household chooses consumption, $\{\widehat{c}_t(\theta_0, s^t)\}$, bond holdings, $\left\{\widehat{b}_t(\theta_0, s^t)\right\}$, and share holdings, $\{\sigma_t(\theta_0, s^t)\}$, to maximize lifetime utility, as in eq. $(*)$, subject to a budget constraint

$$\begin{aligned} & \widehat{c}_t\left(s^t\right) + \frac{\widehat{b}_t\left(s^t\right)}{R_t\left(z^t\right)} + \sigma_t\left(s^t\right)\widehat{v}_t\left(z^t\right) \\ \leq \quad & \eta\left(y_t\right) + \frac{\widehat{b}_{t-1}\left(s^{t-1}\right)}{\lambda\left(z_t\right)} + \sigma_{t-1}\left(s^{t-1}\right)\left[\widehat{v}_t\left(z^t\right) + \alpha\right] \end{aligned}$$

and one of the two borrowing constraints

$$\begin{aligned} \frac{\widehat{b}_t\left(s^t\right)}{R_t\left(z^t\right)} + \sigma_t\left(s^t\right)\widehat{v}_t\left(z^t\right) & \geq \widehat{K}_t\left(y^t\right) \\ \frac{\widehat{b}_t\left(s^t\right)}{\lambda\left(z_{t+1}\right)} + \sigma_t\left(s^t\right)\left[\widehat{v}_{t+1}\left(z^{t+1}\right) + \alpha\right] & \geq \widehat{M}_t\left(y^t\right) \quad \forall z_{t+1} \end{aligned}$$

Equivalence Result II.

Theorem The equilibrium of the stationary Bewley model can be mapped into equilibrium of the Bond model with growth with

$$
\begin{aligned}
c_t\left(\theta_0, s^t\right) &= \widehat{c}_t\left(\theta_0, y^t\right) e_t\left(z^t\right) \\
\sigma_t^B\left(\theta_0, s^t\right) &= \frac{\widehat{a}_t\left(\theta_0, y^t\right)}{\widehat{v}_{t+1}+\alpha} + \widehat{\sigma}_t\left(\theta_0, y^t\right) \\
v_t\left(z^t\right) &= \widehat{v}_t e_t\left(z^t\right) \\
R_t\left(z^t\right) &= \widehat{R}_t \frac{\sum_{z_{t+1}} \phi\left(z_{t+1}\right) \lambda\left(z_{t+1}\right)^{1-\gamma}}{\sum_{z_{t+1}} \phi\left(z_{t+1}\right) \lambda\left(z_{t+1}\right)^{-\gamma}}
\end{aligned}
$$

and bond holdings given by $b_t\left(\theta_0, s^t\right) = 0$.

Why is $b_t(\theta_0, s^t) = 0$?

Use the equivalence between the Bewley model and Bond model to rewrite the budget constraint in the Bond model as

$$\widehat{c}_t\left(y^t\right) + \frac{\widehat{b}_t\left(y^t\right)}{R_t} + \sigma_t\left(y^t\right)\widehat{v}_t \leq \eta\left(y_t\right) + \frac{\widehat{b}_{t-1}\left(y^{t-1}\right)}{\lambda\left(z_t\right)} + \sigma_{t-1}\left(y^{t-1}\right)\left[\widehat{v}_t + \alpha\right]$$

Portfolio Irrelevance Result Implies Risk Premium Irrelevance Result

- Risk premia are identical in the representative agent model and the Arrow as well as the Bond model.
- Risk-free rate is lower in the Bond and Arrow models than in the representative agent model.