If You're So Smart Why Aren't You Rich? Belief Selection in Complete and Incomplete Markets

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Motivation

Market selection hypothesis: markets favor traders with more accurate beliefs

Current studies are inconclusive

- Sandroni (2000)
- DeLong, Shleifer, Summers and Waldmann (1990, 1991)

Need for a complete unifying approach

Preview of Results

- Complete markets: market selection hypothesis holds.
  - 1. Rational traders vs. "the others": survival index
  - 2. Bayesian traders vs. "the others" (size of the support of their beliefs, patience)
- Incomplete markets: market selection hypothesis may fail.
  In particular, an overly optimistic/pessimistic agent may come to dominate the market and the rational agents vanish
- The key to understanding selection for or against traders with rational expectations is Pareto optimality.

Basics

- The possible states at each date form a finite set $\{1, ..., S\}$.
- The set of all sequences of states is $\sum$ with a representative sequence $\sigma = (\sigma_0, ...)$ (also called a path).
- $\sigma^t = (\sigma_0, ..., \sigma_t)$ denotes the partial history through date t of the path σ
- $\mathbf{1}_t^s(\sigma) = \begin{cases} 1 & \text{if } \sigma_t = s \\ 0 & \text{otherwise} \end{cases}$
- Consumer i's endowment is $e_t^i(\sigma)$.
- Consumer i's utility is given by

$$U^i(c) = E_{p^i}\left\{\sum_{t=0}^{\infty} \left(\beta^i\right)^t u^i(c_t(\sigma))\right\}$$

where p^i is the forecast distribution (as opposite to the true distribution p)

Axioms

- **Axiom 1. (i)** The payoff functions $u^i : R_+ \to (-\infty, \infty)$ are C^1, strictly concave and strictly monotonic. **(ii)** The payoff functions satisfy an Inada condition at 0, that is $u^{i\prime}(c) \to \infty$ as $c \downarrow 0$.
- **Axiom 2.** We have $\infty > F = \sup_{t,\sigma} \sum_i e_t^i(\sigma) \geq \inf_{t,\sigma} \sum_i e_t^i(\sigma) = f > 0.$
- **Axiom 3**. For all consumers i, all dates t, and all paths σ, if $p_t(\sigma) > 0$, then $p_t^i(\sigma) > 0$.

Setting-up the Problem

If $c^* = \left(c^{1*}, ..., c^{I*}\right)$ is a Pareto-optimal allocation, then there is a vector of welfare weights $(\lambda_1, ..., \lambda_I) >> 0$ such that c^* solves the problem

$$\max_{\left(c^1, ..., c^I\right)} \sum_i \lambda_i U^i(c)$$

$$\text{such that } \sum_i c^i - e \leq 0$$

$$c^i_t(\sigma) \geq 0 \quad \forall t, \sigma$$

- There is a shadow price $\eta_t(\sigma) > 0$ such that if $p^i_t(\sigma) > 0$ then $\forall t, \sigma$

$$\lambda_i \beta_i^t u^{i'}\left(c^i_t(\sigma)\right) p^i_t(\sigma) - \eta_t(\sigma) = 0$$

- If $p^i_t(\sigma) = 0$ then $c^i_t(\sigma) = 0$.

"Solution" Method

- **Definition 1**. Trader i *vanishes* on path σ if and only if $\lim_t c_t^i(\sigma) = 0$. She *survives* on path σ if and only if $\limsup_t c_t^i(\sigma) > 0$.

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- From the FOCs for agent i and j determine the ratio of their marginal utilities of consumption along a given path of states

$$\frac{u^{i\prime}\left(c_t^i(\sigma)\right)}{u^{j\prime}\left(c_t^j(\sigma)\right)} = \frac{\lambda_j}{\lambda_i}\left(\frac{\beta_j}{\beta_i}\right)^t \frac{p_t^j(\sigma)}{p_t^i(\sigma)}$$

Result 1: Survival Index

- Since $p_t^i(\sigma) = \prod_{\tau=0}^{t} p^i(\sigma_\tau | \mathcal{F}_{\tau-1}) = \prod_{\tau=0}^{t} \prod_{s \in S} p^i(s | \mathcal{F}_{\tau-1})^{\mathbf{1}_\tau^s(\sigma)}$ then $\log \frac{u^{i\prime}(c_t^i(\sigma))}{u^{j\prime}(c_t^j(\sigma))} = \log \frac{\lambda_j}{\lambda_i} + t \log \frac{\beta_j}{\beta_i} + \sum_{\tau=0}^{t} \sum_{s \in S} \mathbf{1}_\tau^s(\sigma) \log p^j(s | \mathcal{F}_{\tau-1}) - \sum_{\tau=0}^{t} \sum_{s \in S} \mathbf{1}_\tau^s(\sigma) \log p^i(s | \mathcal{F}_{\tau-1})$.

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- Then $\frac{1}{t}\log \frac{u^{i'}\left(c_t^i(\sigma)\right)}{u^{j'}\left(c_t^j(\sigma)\right)} = \frac{1}{t}\log\frac{\lambda_j}{\lambda_i} + \log\frac{\beta_j}{\beta_i} + \frac{1}{t}\sum_{\tau=0}^{t}\sum_{s\in S}\mathbf{1}_\tau^s(\sigma)\log\frac{p(s|\mathcal{F}_{\tau-1})}{p^i(s|\mathcal{F}_{\tau-1})} - \frac{1}{t}\sum_{\tau=0}^{t}\sum_{s\in S}\mathbf{1}_\tau^s(\sigma)\log\frac{p(s|\mathcal{F}_{\tau-1})}{p^j(s|\mathcal{F}_{\tau-1})}$.

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- Then $\frac{1}{t} \log \frac{u^{i'}\left(c_t^i(\sigma)\right)}{u^{j'}\left(c_t^j(\sigma)\right)} = \frac{1}{t} \log \frac{\lambda_j}{\lambda_i} + \log \frac{\beta_j}{\beta_i} + \frac{1}{t} \sum_{\tau=0}^{t} \sum_{s\in S} \mathbf{1}_\tau^s(\sigma) \log \frac{p(s|\mathcal{F}_{\tau-1})}{p^i(s|\mathcal{F}_{\tau-1})} - \frac{1}{t} \sum_{\tau=0}^{t} \sum_{s\in S} \mathbf{1}_\tau^s(\sigma) \log \frac{p(s|\mathcal{F}_{\tau-1})}{p^j(s|\mathcal{F}_{\tau-1})}$.
- Define the relative entropy of probability distribution q with respect to distribution p as $\mathcal{I}_p(q) = \sum_{s\in S} p(s) \log \frac{p(s)}{q(s)}$.

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- Then $\frac{1}{t}\log\frac{u^{i'}(c_t^i(\sigma))}{u^{j'}(c_t^j(\sigma))} = \frac{1}{t}\log\frac{\lambda_j}{\lambda_i} + \log\frac{\beta_j}{\beta_i} + \frac{1}{t}\sum_{\tau=0}^{t}\sum_{s\in S}\mathbf{1}_\tau^s(\sigma)\log\frac{p(s|\mathcal{F}_{\tau-1})}{p^i(s|\mathcal{F}_{\tau-1})} - \frac{1}{t}\sum_{\tau=0}^{t}\sum_{s\in S}\mathbf{1}_\tau^s(\sigma)\log\frac{p(s|\mathcal{F}_{\tau-1})}{p^j(s|\mathcal{F}_{\tau-1})}$.
- Define the relative entropy of probability distribution q with respect to distribution p as $\mathcal{I}_p(q) = \sum_{s\in S} p(s)\log\frac{p(s)}{q(s)}$.
- Then with *iid* beliefs and $\frac{1}{t}\log\frac{u^{i'}(c_t^i(\sigma))}{u^{j'}(c_t^j(\sigma))} \longrightarrow \left(\log\beta_j - \mathcal{I}_p(p^j)\right) - \left(\log\beta_i - \mathcal{I}_p(p^i)\right)$. The expression $\kappa_i = \log\beta_i - \mathcal{I}_p(p^i)$ is a *survival index*.

Risk Aversion and Rate of Convergence

Example. Suppose that traders i and j each have utility function $u^i(c) = \alpha^{-1} c^{\alpha}$ for $\alpha < 1$. In this case $\frac{1}{t} \log \frac{c_t^i(\sigma)^{\alpha-1}}{c_t^j(\sigma)^{\alpha-1}} \longrightarrow \kappa_j - \kappa_i$.

Hence, $\frac{c_t^i(\sigma)}{c_t^j(\sigma)} \approx \exp\left(\frac{-t\left(\kappa_j - \kappa_i\right)}{1-\alpha}\right)$.

Result 2: A Bayesian Is Sure He Will Survive

Proof. Start from $\frac{u^{i'}\left(c_t^\theta(\sigma)\right)}{u^{j'}\left(c_t^i(\sigma)\right)} = \frac{\lambda_j}{\lambda_i}\frac{p_t^i(\sigma)}{p_t^\theta(\sigma)}$. The first ratio on the RHS is a constant. The second ratio is a nonnegative martingale with mean 1 under p^θ so it converges almost surely. Then, the Lemma 1 condition for vanishing fails to hold. QED.

Result 3: Any Trader Who Survives Is Not Too Different From a Bayesian.

Proof. Start from $\frac{\lambda_j}{\lambda_i}\frac{u^{i'}\left(c_t^{\theta}(\sigma)\right)}{u^{j'}\left(c_t^{j}(\sigma)\right)} = \frac{p_t^j(\sigma)}{p_t^{\theta}(\sigma)}$. By contradiction suppose that the beliefs are not mutually absolutely continuous. Suppose that there is a measurable subset A of V such that $p_t^{\theta}(\sigma) = 0$ and $p_t^{j}(\sigma) > 0$. Then, $\frac{p_t^j(\sigma)}{p_t^{\theta}(\sigma)} \to \infty$ for all $\sigma \subset A$. QED.

Result 4: Market Selects Among Bayesians According to the Size of the Support of Their Beliefs

Proof is quite elaborate.
Intuition is quite straightforward: the larger the size of the support of beliefs the slower the learning process.

Result 5: The Effect of Belief Differences Works Much Slower Than the Effect of Discount Factor Differences

Intuition: Think of two traders

- i has a lower-dimensional prior but low β^i
- j has a higher-dimensional prior but $\beta^j \to 1$

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Consequently, among Bayesians whose prior belief supports contain the truth the survivors are always among the **most patient ones.**

Incomplete Markets 1: Undue Optimism or Pessimism

- Two traders with CRRA utility, one seller with logarithmic utility.
- There is a single asset that pays off
$$R_t(\sigma) = \begin{cases} \left(1 + \left(\frac{1}{2}\right)^t\right) & \text{if } \sigma_t = s_1 \\ k\left(1 + \left(\frac{1}{2}\right)^t\right) & \text{if } \sigma_t = s_2 \end{cases} \quad \text{where } k > 1.$$
- The truth has s_1 happening at every t. So are the beliefs of trader 1, while trader 2 incorrectly believes that s_1 and s_2 always occur with equal probability.
- The equilibrium price of an asset is $q_t(\sigma) = \frac{1}{2}\left(1 + \left(\frac{1}{2}\right)^t\right)$.
- Trader 1's wealth is $\left(\frac{1}{2}\right)^{t-1}$ and he consumes $\frac{3}{4}$ of this wealth while trader 2's wealth is 1 at each date and at each date he consumes $\frac{1}{2}$ of his wealth.
- At each date t trader 1 believes that the rate of return on asset is 2 while trader 2 believes it is 2 with prob. $\frac{1}{2}$ and $2k > 2$ with prob. $\frac{1}{2}$.

Incomplete Markets 2: Failure of Relative Entropy

- Assets pay off according to: $R_t^1(\sigma) = \begin{cases} 1 \text{ if } \sigma_t = s_1 \\ 0 \text{ if } \sigma_t = s_2, s_3 \end{cases}$

  $R_t^2(\sigma) = \begin{cases} 0 \text{ if } \sigma_t = s_1 \\ 1 \text{ if } \sigma_t = s_2, s_3 \end{cases}$
- Traders' endowment: $e_t^i = \begin{cases} \frac{1}{2} \text{ if } t = 1 \\ 0 \text{ if } t > 1 \end{cases}$ for $i = 1, 2$

  $e_t^3 = \begin{cases} 0 \text{ if } t = 1 \\ 1 \text{ if } t > 1 \end{cases}$
- The state probabilities and beliefs are given in the table ($\varepsilon > 0$)

	States		
	s_1	s_2	s_3
Truth	$\frac{1}{2}$	$\frac{1}{2} - \varepsilon$	ε
Trader 1	$\frac{1}{2}$	$\frac{1}{2} - \varepsilon$	ε
Trader 2	$\frac{1}{2}$	ε	$\frac{1}{2} - \varepsilon$
Trader 3	$\frac{1}{2}$	$\frac{1}{2} - \varepsilon$	ε

Failure of Relative Entropy cont'd

- The relative entropies of beliefs with respect to the truth are given as
$\mathcal{I}_p\left(p^1\right)=\frac{1}{2}\log 1+\left(\frac{1}{2}-\varepsilon\right)\log 1+\varepsilon\log 1=0$
$\mathcal{I}_p\left(p^2\right)=\frac{1}{2}\log 1+\left(\frac{1}{2}-\varepsilon\right)\log\frac{\left(\frac{1}{2}-\varepsilon\right)}{\varepsilon}+\varepsilon\log\frac{\varepsilon}{\left(\frac{1}{2}-\varepsilon\right)}\neq 0.$
- But since $E_t^i\left(R_t^1\left(\sigma\right)\right)=E_t^i\left(R_t^2\left(\sigma\right)\right)=\frac{1}{2}$ for $i=1,2,3$ and $\forall t,\sigma$ in equilibrium both traders hold identical amounts of both assets. The distribution of wealth between traders 1 and 2 remains unchanged.

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- If you live in the complete-markets world smart people are the ones who will be rich unless...
- ... they learn too slowly or...
- ... they are not patient enough
- If the markets are incomplete it is possible that the rich are not necessarily smart

Brief Discussion

- Disconnect between utility maximization and survival
- Behavior of prices *versus* survival concept
- Sequential rationality ?
- A model of selection/evolution with infinitely lived agents ?