Which Moments to Match?

Gallant and Tauchen

Econometric Theory 1996

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MOTIVATION

- Suppose that you have a structural model for which:
  1. you need to choose parameter values, and
  2. the likelihood function is difficult to compute.
- What do you do?
  - Often people choose the parameter values to match a few low order moments of the data.
  - This paper proposes an alternative.

THE IDEA

- **Idea:** use the equations that you would use for MLE as the basis for moments for GMM.
  - Set their expected value equal to zero for GMM.
- How can we compute these expected values?
  - To compute the derivatives of the log-likelihood we use an auxiliary model of the data for which we have these derivatives in closed form.
  - We compute their expected value by simulation, using the probability distribution implied by our structural model.

Setup

Observed data: $\{\tilde{y}_t, \tilde{x}_t\}_{t=1}^{n}$

Model: $\left\{p_1(x_1, \rho), \{p_t(y_t|x_t, \rho)\}_{t=1}^{n}\right\}$

Auxiliary model: $\left\{f_1(x_1, \theta), \{f_t(y_t|x_t, \theta)\}_{t=1}^{n}\right\}$

Assumption: Given the correct parameter values, the model is the true DGP.

GMM MOMENT EQUATIONS

- **Define** $\tilde{\theta}_n$ to be the MLE of θ:

$$\tilde{\theta}_n \equiv \arg\max_{\theta \in \Theta} \frac{1}{n} \sum_{t=1}^{n} \ln f_t(\tilde{y}_t | \tilde{x}_t, \theta).$$

- Proposed **GMM moment equation**:

$$m_n(\rho, \tilde{\theta}_n) = \frac{1}{n} \sum_{t=1}^{n} \int \dots \int \frac{\partial}{\partial \theta} \ln f_t(y_t | x_t, \tilde{\theta}_n) \times \prod_{\tau=1}^{n} p_\tau(y_\tau | x_\tau, \rho) dy_\tau p_1(x_1 | \rho) dx_1.$$

- Remarks
  - Intuition
  - We can specialize this.

IMPLEMENTATION BY SIMULATION

- To evaluate the moment equation we can use simulation: N samples of size n.
  - N needs to be large.

- **Moment equation:**

$$m_n(\rho, \tilde{\theta}_n) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{N} \sum_{\tau=1}^{N} \frac{\partial}{\partial \theta} \ln f_t(\hat{y}_{t\tau} | \hat{x}_{t\tau}, \tilde{\theta}_n).$$

ESTIMATION

- Under regularity conditions on p_t and f_t

$$\sqrt{n} m_n(\rho^0, \tilde{\theta}_n) \approx N(0, \mathcal{J}_n^0).$$

- The **GMM estimator with an efficient weighting matrix** is

$$\hat{\rho}_n = \arg\min_{\rho \in R} m_n'(\rho, \tilde{\theta}_n)(\tilde{\mathcal{J}}_n)^{-1} m_n(\rho, \tilde{\theta}_n),$$

where $\tilde{\mathcal{J}}_n$ satisfies

$$\lim_{n \to \infty} (\tilde{\mathcal{J}}_n - \mathcal{J}_n^0) = 0.$$

WEIGHTING MATRIX

- How you estimate the weighting matrix depends on what assumptions you make about the auxiliary model.
- Possible assumptions, in increasing strength, are:
  1. $\exists\ \theta^0$ such that
  $$\int \ldots \int \frac{\partial}{\partial\theta} \ln f_t(y_t|x_t,\theta^0) \times \prod_{\tau=1}^{n} p_\tau(y_\tau|x_\tau,\rho^0) dy_\tau p_1(x_1|\rho^0) dx_1 = 0$$
  for every $t \leq n$.
  2. The auxiliary model contains the true model: $\exists\ \theta^0$ such that
  $$\begin{aligned} p_t(y_t|x_t,\rho^0) &= f_t(y_t|x_t,\theta^0) \quad \forall\ t = 1,\ldots,n \\ p_1(x_1|\rho^0) &= f_1(x_1|\theta^0). \end{aligned}$$

Weighting Matrix cont.

3. There exists and open neighborhood, R^0, of ρ^0 such that there's a C^2 mapping $g : R^0 \to \Theta$ for which

$$p_t(y_t|x_t,\rho) = f_t(y_t|x_t, g(\rho)) \quad \forall\ t = 1,\ldots,n,$$
$$p_1(x_1|\rho) = f_1(x_1|g(\rho))$$

for all $\rho \in R^0$.

- Remark: Choosing the auxiliary model.

ASYMPTOTIC DISTRIBUTION

Theorem: Under any of assumptions (1), (2) or (3), the GMM estimator $\hat{\rho}_n$ has an asymptotically normal distribution:

$$\sqrt{n}(\hat{\rho}_n - \rho^0) \approx N(0, V).$$

The variance matrix depends on which assumption is made.

- When Assumption 3 is satisfied the estimator is as efficient as the MLE.

CONSUMPTION-SAVINGS PROBLEM

- A representative agent faces the following problem:

$$\max_{\{c_{t+i},k_{t+1+i}\}_{i=0}^{\infty}} \mathbb{E}_t\left[\frac{1}{1-\gamma}\sum_{i=0}^{\infty}\beta^i c_{t+i}^{1-\gamma} v_{2,t+i}\right]$$

$$\text{s.t. } c_t + k_{t+1} - k_t \leq A k_t^{\alpha} v_{1t}.$$

  - $v_t \equiv (v_{1t}, v_{2t})'$ is a strictly stationary r order Markov process parameterized by δ.
  - State variables: v_t and k_t.
- Euler equation: $c_t^{-\gamma} v_{2t} = \mathbb{E}_t\left[\beta c_{t+1}^{-\gamma} v_{2,t+1}(\alpha A k_{t+1}^{\alpha-1} v_{1,t+1} + 1)\right]$.
- Policy functions: $k_{t+1} = \psi_k(k_t, v_t)$ and $c_t = \psi_c(k_t, v_t)$.

CALIBRATION

- Parameters to calibrate: $\rho = \{\gamma, A, \alpha, \delta\}$.
- Data: $y_t \equiv (c_t, r_{et})'$
  - r_{et} is the excess stock return over the bond return.
- Calculating bond and stock returns:

$$c_t^{-\gamma} v_{2t} = \mathbb{E}_t \left[\beta c_{t+1}^{-\gamma} v_{2,t+1}\right] (1 + r_{b,t+1})$$
$$p_{st} c_t^{-\gamma} v_{2t} = \mathbb{E}_t \left[\beta c_{t+1}^{-\gamma} v_{2,t+1} (p_{s,t+1} + d_{s,t+1})\right].$$

- Auxiliary model for y_t: ARCH model with an AR process for the mean equation.

CALIBRATION PROCEDURE

1. Use the observed data $\{\tilde{y}_t\}_{t=1}^n$ to estimate θ:

$$\tilde{\theta}_n \equiv \arg\max_{\theta \in \Theta} \frac{1}{n} \sum_{t=1}^{n} \ln f_t(\tilde{y}_t | \tilde{x}_t, \theta).$$

2. Simulate N samples of length n for the model and compute the moment equation:

$$m_n(\rho, \tilde{\theta}_n) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{N} \sum_{\tau=1}^{N} \frac{\partial}{\partial \theta} \ln f_t(\hat{y}_{t\tau} | \hat{x}_{t\tau}, \tilde{\theta}_n).$$

3. Compute the weighting matrix and estimate ρ:

$$\hat{\rho}_n = \arg\min_{\rho \in R} m_n'(\rho, \tilde{\theta}_n)(\tilde{\mathcal{J}}_n)^{-1} m_n(\rho, \tilde{\theta}_n),$$