Belief Disagreement and Collateral Constraints

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Motivation (I) - What does the paper do ?

Modifies a stylized general equilibrium model with two features

1. Endogenous borrowing constraints
   Borrowing contracts are collateralized by assets whose value is determined in equilibrium
2. Belief disagreements
   Agents have heterogeneous priors on distribution of aggregate states

Motivation (II) - What does the paper get ?

The interaction of heterogeneous beliefs and agency frictions is conditional on the space of contracts

1. Restricted contract spaces : What kind of belief disagreements matter for prices ?
2. Unrestricted contract spaces : What kind of contracts emerge due to belief disagreements ?

Setup (I) - Preferences and Technology

1. Time : $T \in \{0, 1\}$
2. States : Aggregate uncertainty : $s \in \mathcal{S} = [s^{min}, s^{max}]$
3. Agents / Preferences :
   - Continuum of risk neutral traders who consume in period $T = 1$.
   - Each trader has an endowment n_i in period $T = 0$ and a prior F_i on $\mathcal{S}$
4. Technology : There is a Lucas tree with and a risk free bond (cash) with payoffs $D(s) = s$ and $D(s) = 1$ respectively. [1]

[1] One unit of the tree is held by unmodeled agents who sell it $T = 0$ and consume the proceeds.

Setup (II) - Trading arrangements

Define a borrowing contract : β and the contract space $\mathcal{B}$ as follows

$$\beta = \left\langle \underbrace{\psi(s) \geq 0}_{Promise}, \underbrace{\alpha}_{collateral:asset}, \underbrace{\gamma}_{collateral:cash} \right\rangle$$

$$\mathcal{B} \equiv \left\{ (\psi, \alpha, \gamma) | \psi \in \mathcal{S}^{\mathbb{R}_+} \text{is measurable and bounded and } (\alpha, \gamma) \in \mathbb{R}^2_+ \right\}$$

- Borrowing contracts are traded in anonymous competitive markets at price $q(\beta)$. Combining default and non-default events the payoff on promise β

$$z(s|\beta) = \min\{\alpha s + \gamma, \psi(s)\}$$

- Agents cannot (directly) short the Lucas tree or cash

Restrictions on β yield familiar arrangements like risky debt : $\psi(s) = \psi$, short contract : $\psi(s) = \psi s$

Agent's problem

Given $p, q(\beta)$ Each agent solves

$$\max_{(a_i,m_i)\in\mathbb{R}^2_+;\mu_i^+,\mu_i^-} a_i\mathbb{E}_i[s] + m_i + \mathbb{E}_i\int_{\mathcal{B}} z(s|\beta)d\mu_i^+ - \mathbb{E}_i\int_{\mathcal{B}} z(s|\beta)d\mu_i^- \tag{1}$$

$$pa_i + m_i + \underbrace{\int_{\mathcal{B}} q(\beta)d\mu_i^+}_{lending} = n_i + \underbrace{\int_{\mathcal{B}} q(\beta)d\mu_i^-}_{borrowing} \tag{2a}$$

$$\int_{\mathcal{B}} \alpha d\mu_i^- \leq a_i \tag{2b}$$

$$\int_{\mathcal{B}} \gamma d\mu_i^- \leq m_i \tag{2c}$$

General Equilibrium with Collateralized Borrowing (GECB)

A GECB is a collection of prices $\{p, q(\beta)\}$ and portfolios $(a_i, m_i, \mu_i^+, \mu_i^-)_{i \in I}$ such that the agents optimize and asset, borrowing markets clear

Next steps :

1. Restrict heterogeneity : Two types - optimists and pessimists
2. Comparative statics with restricted contract spaces : standard debt contracts , short contracts
3. Equilibrium in unrestricted contract space

Standard Debt Contracts

Let F_1, F_0 denote the beliefs the optimists and pessimists respectively, the contract space with standard debt contracts is given by

$$\mathcal{B}^D \equiv \{\psi(s) = \psi \in \mathbb{R}_+, 1, 0\} \tag{3}$$

Assumptions :

1. $n_1 < \mathbb{E}_1[s] - s^{min}$ and $n_0 + n_1 > E_1[s]$
2. F_1 dominates F_0 in hazard rate order

$$\frac{f_1(s)}{1 - F_1(s)} < \frac{f_0(s)}{1 - F_0(s)}$$

Principal-Agent problem

For asset prices $p \in (\mathbb{E}_0[s], \mathbb{E}_1[s])$, consider a contracting problem between borrower (optimist) and lender (pessimists).

$$\max_{a_1, \psi, q} a_1 \mathbb{E}_1 \left(s - min(s, \psi) \right) \tag{4}$$

s. t

$$a_1 p = n_1 + a_1 q \tag{5}$$

$$q = \mathbb{E}_0[min(s, \psi)] \tag{6}$$

General Equilibrium with Standard Debt Contracts

Using the solution to the principal agent problem we can construct a general equilibrium with standard debt contracts as follows

- $p^* : a_1(p) = 1$
- $a_1^* = 1, m_1^* = 0$
- $m_0^* = n_0 + n_1 - p^*$
- $q^*(\psi) = \mathbb{E}_0[min(s, \psi)]$
- $\mu_1^+ = \mu_0^- = 0$
- μ_1^- (or μ_0^+) is a Dirac measure on $\psi(p^*) \in \mathcal{B}^D$

Remarks

1. Competitive equilibrium allocates the bargaining power to the optimists (borrowers)

2. Equilibrium is essentially unique

Characterization : Standard Debt Contracts

We get the following FOC from the optimist's problem

$$p = F_0(\psi)\mathbb{E}_0[s|s < \psi] + [1 - F_0(\psi)]\mathbb{E}_1[s|s > \psi] \tag{7}$$

Return on endowment for the optimist $R_1^L(\psi)$ is given by

$$\frac{\mathbb{E}_1[s] - \mathbb{E}_1[min(s, \psi)]}{p - \mathbb{E}_0[min(s, \psi)]}$$

Comparative Statics : Standard Debt Contracts

Consider an equilibrium with standard debt contracts, Now perturb the beliefs to $\tilde{F}_i$ such that $\forall s$

$$\frac{\tilde{f}_1(s)}{1-\tilde{F}_1(s)} \leq \frac{f_1(s)}{1-F_1(s)} \tag{8a}$$

$$\frac{\tilde{f}_0(s)}{1-\tilde{F}_0(s)} \geq \frac{f_0(s)}{1-F_0(s)} \tag{8b}$$

We have the following result

1. Suppose above conditions are satisfied with equality on $s \in (s^{min}, \psi^*)$, asset prices are higher
2. Suppose above conditions are satisfied with equality on $s \in (\psi^*, s^{max})$, asset prices are lower

General Equilibrium with Short Selling

A typical short sale is described by two objects *margin* and *lender's fee*.

1. A trader (pessimist) borrows an asset from the lender (optimist)
2. The borrower sells the asset and raises p dollars from the market
3. The asset borrowing is backed by a gross cash collateral of $(1 + m^s)p$
4. When the asset is returned the lender repays the collateral with a interest $r^{rebate}(< r = 0)$. This spread is the fee for lending the asset

$$\mathcal{B} \equiv \{[\psi(s) = s], 0, \gamma\}$$

where

$$\gamma = (1 + m^s)(1 + r^{rebate})p$$

$$p - q(\gamma) = -r^{rebate}(1 + m^s)p$$

General Equilibrium with Unrestricted Contract Space

GE with unrestricted contract spaces are equivalent to Arrow Debreu economies with solvency constraints
Given a menu of arrow securities [2], the price for the Lucas tree, the traders solve

$$\max_{a_i, m_i, z_i(s)} \int_S [a_i s + m_i + z_i(s)] dF_i(s) \tag{9}$$

subject to

$$a_i p + m_i + \int z_i(s) q^{AD}(s) ds \leq n_i$$

$$a_i s + m_i + z_i(s) \geq 0 \quad \forall s \in \mathcal{S}$$

[2] There are two degrees of freedom for the portfolio due to redundant assets - Lucas tree and the risk free bond. This is resolved by making the optimist hold the entire supply of the tree and none of the risk free asset

General Equilibrium with Unrestricted Contract Space

Given an AD equilibrium $\{p, q^{AD}, (a_i, m_i, z_i)\}$ we can implement it in a General Equilibrium with Collateralized Borrowing with

1. Contracts : $\beta_i = \langle \psi_i(s) = max(0, -z_i(s)), \alpha_i = a_i, \gamma_i = m_i \rangle$
2. Prices: $q(\beta) = \int min[\alpha s + \gamma, \psi(s)] q^A D(s)$ and p
3. Portfolios : (a_i, m_i) and a Dirac measure μ_i^- that puts weight on β_i

Remark :
The non negativity of consumption in the AD equilibrium and the richness of the contract space implies that the AD eq is feasible with borrowing contracts.

Characterizing a AD economy : Asset Prices

Suppose F_1 dominates F_0 is likelihood sense (or they satisfy MLRP), the pricing kernel of AD equilibrium is given by

$$q^{AD}(s) = max\left(\frac{f_0(s)}{R_0}, \frac{f_1(s)}{R_1}\right)$$

Where R_i (Agent i's return on wealth or the Lagrange multiplier on respective budget constraints)

Remarks
As before optimism about relative likelihood of downside states has no effect on the price of the Lucas tree

$$p = \int_{\mathcal{S}} sq^A D(s) ds$$

CDOs and CDS ?

Suppose F_1 dominates F_0 is likelihood sense (or they satisfy MLRP), the portfolios allocation is given by the following

$$z_1(s) = -s\mathbb{I}_{s<\bar{s}} + m_0\mathbb{I}_{s>\bar{s}}$$

$$z_0(s) = s\mathbb{I}_{s<\bar{s}} - m_0\mathbb{I}_{s>\bar{s}}$$

The asset is tranched so that the optimist hold the payoff in $s > \bar{s}$

while selling it to the pessimist in $s < \bar{s}$

Extra

1. Efficiency
2. Idiosyncratic risks
3. Measuring types of belief dispersions
4. Long run dynamics and survival
5. Endogenity of beliefs